# Computation of Koszul homology and application to partial differential systems

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# I. Motivations

- ▷ Algebraic analysis and formal theory of PDEs
- ▷ Formal integrability
- ▷ 2-acyclicity and Koszul homology

# II. Computation of Koszul homology

- ▷ Symbol module
- ▷ Koszul homology of the symbol module
- ▷ 2-acyclicity criterion

# III. Examples

- ▷ Constant coefficients
- ▷ Parametric coefficients
- ▷ Polynomial coefficients

# IV. Conclusion and perspectives

# I. MOTIVATIONS

Consider a PDEs system

$$\Sigma : \quad F^{\rho}\left(x^i, \frac{\partial^{|\mu|} u^j}{\partial x^{\mu}}\right) = 0$$

➔ $x^1, \cdots, x^n$, $u^1, \cdots, u^p$: (in)dependent variables

➔ $F^1, \cdots, F^q$: equations of the system

**Study of $\Sigma$**

**Analytical approach**

*Objective:* prove existence and uniqueness of solutions

➔ what type of solution? (classical, weak, distributions)

*Algorithmic side:* approximate solutions by numerical schemes

**Algebraic approach**

*Objective:* investigate algebraic structure of the solution space

➔ symbolic solutions (formal power series, Laplace inversion)

*Algorithmic side:* compute exact solutions

**Algebraic analysis:**

Apply tools from algebraic geometry to PDEs
($D$-modules, homological algebra, sheaf theory)

**Module of a linear system $Ru = 0$:** $\mathcal{M} := \text{coker}(.R) = \mathcal{D}^{1\times p}/\mathcal{D}^{1\times q}R$

- ➔ $R \in \mathcal{D}^{q\times p}$, $\mathcal{D}$: a ring of PD operators
- ➔ $u \in \mathcal{F}^p$, $\mathcal{F}$: a functional space (left $\mathcal{D}$-module)
- ➔ $\mathcal{M} \leftrightarrow \mathcal{O} := \mathbb{K}[x_1, \cdots, x_p]/I(f_1, \cdots, f_q)$ from algebraic geometry

**Malgrange's iso.:** $\hom_{\mathcal{D}}(\mathcal{M}, \mathcal{F}) \simeq \ker_{\mathcal{F}}(R.)$ (*i.e.*, $\mathcal{F}$-points $\leftrightarrow$ $\mathcal{F}$-solutions)

- ➔ investigate $(\Sigma : Ru = 0)$ by means of properties of $\mathcal{M}$

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## Example: module of a system, Malgrange's iso.

**System:** 2 unknowns functions $u^1, u^2 \in \mathcal{F}$ and 3 equations

$$\begin{cases} \partial_1 u^1 =_{\mathcal{F}} 0 \\ \frac{1}{2}\left(\partial_2 u^1 + \partial_1 u^2\right) =_{\mathcal{F}} 0 \\ \partial_2 u^2 =_{\mathcal{F}} 0 \end{cases} \iff Ru = \begin{pmatrix} \partial_1 & 0 \\ \frac{1}{2}\partial_2 & \frac{1}{2}\partial_1 \\ 0 & \partial_2 \end{pmatrix} \begin{pmatrix} u^1 \\ u^2 \end{pmatrix} =_{\mathcal{F}} 0$$

**Corresponding module:** $\mathcal{D} := \mathbb{Q}[\partial_1, \partial_2]$ and $\mathcal{M} := \mathcal{D}^{1\times 2}/\mathcal{D}^{1\times 3} R$

➔ $\mathcal{M}$ has 2 generators $m_1, m_2$ subject to 3 relations

$$\partial_1.m_1 =_{\mathcal{M}} 0 \qquad \frac{1}{2}\partial_2.m_1 + \frac{1}{2}\partial_1.m_2 =_{\mathcal{M}} 0 \qquad \partial_2.m_2 =_{\mathcal{M}} 0$$

**Malgrange's iso. illustration:** given $\varphi : \mathcal{M} \to \mathcal{F}$ and letting $u^j := \varphi(m_j)$, then $Ru =_{\mathcal{F}} 0$:

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**Formal theory:** solutions are formal power series

*Procedure for constructing solutions:*

- find relations between the coefficients of the Taylor development
  ➔ candidate for the general solution
- formulate effective criteria for checking validity of the candidate
- deduce the general solution and the dimension of the solution space

**Formalism and computation**

## Differential geometry

- jet bundle of degree $d$: $J_d \to X$
  ➔ $\left(x^i, u^j{}_\mu \leftrightarrow \text{Taylor coeff.}\right) \in J_d$
  ➔ $\Sigma_d = \left(F^\rho\left(x^i, u^j{}_\mu\right) = 0\right) \subseteq J_d$
- differentiation of equations
  ➔ prolongation: $\Sigma_{d+1} \subseteq J_{d+1} \to X$

## Differential algebra

- differential elimination
  ➔ express $u^j{}_\mu$ = lower terms
  ➔ compute integrability conditions
- homological algebra
  ➔ validate integrability conditions

## Formal integrability

**Def.:** A system $\Sigma_d$ (of degree $d$) is **formally integrable** if $\forall m, r \geq 0$:

$$\Sigma_{m+r+d} := J_{m+r}(\Sigma) \cap J_{m+r+d} \twoheadrightarrow \Sigma_{m+d} := J_m(\Sigma) \cap J_{m+d} \text{ (+ regularity conditions)}$$

**Intuition:** differentiations of degrees $m + r$ of $F^\rho$'s

$$\frac{\partial^{|\mu|} F^\rho}{\partial x^\mu}$$

do not bring new integrability conditions of degrees $\leq m + d$

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## Example with one independent variable

$$\Sigma_1 = (\dot{u} = u) \subseteq J_1 \simeq \mathbb{R}^3 \to X \simeq \mathbb{R},\ (t, u, \dot{u}) \mapsto t$$

**Differential elimination:** $u^{(m+r+1)} \to u^{(m+r)} \to \cdots \to u^{(m+1)} \to \cdots \to \dot{u} \to u \quad \forall m, r \geq 0$

**New integrability conditions:** involve $u^{(m+2)}, \cdots, u^{(m+r)}$ ➔ $\Sigma_1$ is formally integrable

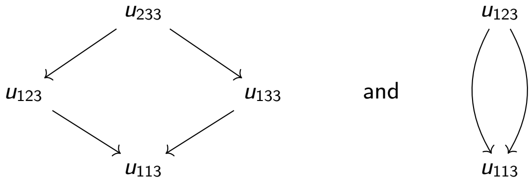
**Solution space:** 1-dimensional space generated by $e^{t-t_0}$ $(\Sigma_\infty := \lim(\cdots \to \Sigma_2 \to \Sigma_1 \to \Sigma))$

$$\Sigma_\infty \ni (t_0, u, \dot{u}, \ddot{u}, \cdots) = (t_0, u, u, u, \cdots) \leftrightarrow u + u(t-t_0) + \frac{u}{2}(t-t_0)^2 + \frac{u}{3!}(t-t_0)^3 + \cdots$$

## Example with three independent variables

$$\Sigma_2 = (u_{33} = u_{23} = u_{13},\ u_{12} = u_{11}) \subseteq J_2 \to X \simeq \mathbb{R}^3,\ (x^i, u, u_i, u_{ij}) \mapsto (x^i)$$

**Differential elimination:** $u_{33} \to u_{13}$, $u_{23} \to u_{13}$ and $u_{12} \to u_{11}$ induce




**New integrability condition:** $\emptyset$ ➔ $\Sigma_2$ is formally integrable

**Solution space:** FPSs obtained from "parametric derivatives":

$$u(x^i) = u + u_1 y^1 + u_2 y^2 + u_3 y^3 \qquad (y^i := (x^i - x_0{}^i))$$
$$+ \frac{u_{11}}{2}(y^1)^2 + u_{11} y^1 y^2 + u_{13} y^1 y^3 + \frac{u_{22}}{2}(y^2)^2 + u_{13} y^2 y^3 + \frac{u_{13}}{2}(y^3)^2$$
$$+ \dots$$

➔ corresponds to $(x_0{}^i, u_i, u_{11}, u_{11}, u_{13}, u_{22}, u_{13}, u_{13}, \cdots) \in \Sigma_\infty$

## Janet example (not formally integrable)

$$\Sigma_2 = (u_{33} = x^2 u_{11},\ u_{22} = 0) \subseteq J_2 \to X \simeq \mathbb{R}^3$$

**Differential elimination:** $u_{33} \to x^2 u_{11}$ and $u_{22} \to 0$ induce

$$u_{2233} \to x^2 u_{1122} + 2u_{112} \to 2u_{112}, \qquad u_{2233} \to 0$$

**New integrability condition:** $u_{112} = 0$ in $\Sigma_4$ but $u_{112}$ is free in $\Sigma_3$

➔ $\Sigma_4 \to \Sigma_3$ is not onto : $\Sigma_2$ is not formally integrable

**Solution space:** ???

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➔ $\Sigma_4 \to \Sigma_3$ **is not onto**: $\Sigma_2$ is not formally integrable

**Solution space: ???**

### Integrability defect and homology

**Question:** what type of finite test for proving formal integrability?

➔ based on differential elimination (*S*-**polynomial like criterion**)

**Fact:** $H^2$ Spencer cohomology of the (comodule) symbol does not vanish for the Janet example

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**Theorem (Goldschmidt).** Assume that **the symbol of $\Sigma_d$ is 2-acyclic**. If $\Sigma_{d+1}$ is a fibered manifold and if $\Sigma_{d+1} \to \Sigma_d$ is onto, then $\Sigma_d$ is formally integrable.

**Theorem (Serre).** Spencer and Koszul complexes are dual to each other.

➔ 2-acyclicity $\Leftrightarrow$ $H_2$ **Koszul homology** of the symbol of $\Sigma_d$ vanishes

## Proving formal integrability

**Procedure:**

- prove 2-acyclicity ➔ **how?**
- prove that $\Sigma_{d+1}$ is a fibered manifold ➔ rank computation
- prove that $\Sigma_{d+1} \to \Sigma_d$ is onto ➔ differential elimination

**If fail:** apply a finite prolongation/projection method

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**Our contribution:**

**compute $H_2$ Koszul homology** of **linear systems**

➔ using OreMorphisms and OreModules **Maple** packages

# II. COMPUTATION OF KOSZUL HOMOLOGY

## Symbol module

**Background:** $(\Sigma_d : Ru = 0)$, $R \in \mathcal{D}^{q\times p}$, $\mathcal{D} = \mathcal{B}\langle \partial_1, \cdots, \partial_n \rangle$

- ➔ $\mathcal{B}$ : a commutative differential ring, equipped with derivations $\delta_i$'s
- ➔ $\mathcal{D}$: generated by $\mathcal{B}$ and $\partial_i$'s, subject to relations $\partial_i \partial_j = \partial_j \partial_i$, $\partial_i b = b\partial_i + \delta_i(b)$
- ➔ $\mathcal{D}$ is a filtered ring: $\partial_i b = b\partial_i + \delta_i(b)$ is not a homogeneous equation

**Informal def.:** the symbol module is the module of the top-degree part of $\Sigma_d$

- ➔ cancels low derivatives in the equations  ($e.g.$, $\partial_0 u = \partial_{11} u$ ➔ $0 = \partial_{11} u$)

**Construction:** based on the graded ring of $\mathcal{D} = \bigcup \mathcal{D}_m := \bigcup \mathcal{B}\langle \partial_1, \cdots, \partial_n \rangle_m$

- ➔ $\mathcal{A} := \mathrm{gr}(\mathcal{D}) = \bigoplus (\mathcal{D}_m/\mathcal{D}_{m-1})$, $\sigma_m : \mathcal{D}_m \to \mathcal{D}_m/\mathcal{D}_{m-1}$, $\chi_i := \sigma_1(\partial_i)$
- ➔ $\mathcal{A} \simeq \mathcal{B}[\chi_1, \cdots, \chi_n]$ : $\chi_i b = \sigma_1(\partial_i b) = \sigma_1(b\partial_i + \delta_i(b)) = \sigma_1(b\partial_i) = b\chi_i$
- ➔ the symbol module is the graded module of $\sigma_d(R) := \left(\sigma_d\left({R^i}_j\right)\right) \in \mathcal{A}^{q\times p}$:

$$\mathrm{gr}\left(\mathcal{M}\right) := \mathcal{A}(0)^{1\times p}/\mathcal{A}(-d)^{1\times q}\sigma_d(R) \qquad \text{(shifts will be omitted)}$$

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## Examples (with $p = 1$ unknown function)

**Constant coefficients:** $(\Sigma_2 : \ u_{33} = u_{23} = u_{13}, \ u_{12} = u_{11})$

- $R = \left(\partial_3^2 - \partial_1\partial_3 \quad \partial_2\partial_3 - \partial_1\partial_3 \quad \partial_1\partial_2 - \partial_1^2\right)^T$
- $\mathcal{D} = \mathbb{Q}[\partial_1, \partial_2, \partial_3] \simeq \mathcal{A} = \mathbb{Q}[\chi_1, \chi_2, \chi_3]$

  ($\mathcal{B} = \mathbb{Q}$ has trivial derivations: $\partial_i q = q\partial_i$)

➔ $\mathrm{gr}\,(\mathcal{M}) = \mathcal{A}/\mathcal{A}^{1\times 3}\left(\chi_3^2 - \chi_1\chi_3 \quad \chi_2\chi_3 - \chi_1\chi_3 \quad \chi_1\chi_2 - \chi_1^2\right)^T$

**Parametric coefficients:** $(\Sigma_2 : \ u_{33} = u_{22} = u_{13} = u_{12} = 0, \ u_{23} = \alpha u_{11})$

- $R = \left(\partial_3^2 \quad \partial_2\partial_3 - \alpha\partial_1^2 \quad \partial_2^2 \quad \partial_1\partial_3 \quad \partial_1\partial_2\right)^T$
- $\mathcal{D} = \mathbb{Q}[\alpha][\partial_1, \partial_2, \partial_3] \simeq \mathcal{A} = \mathbb{Q}[\alpha][\chi_1, \chi_2, \chi_3]$

  ($\mathcal{B} = \mathbb{Q}[\alpha]$ has trivial derivations: $\partial_i p(\alpha) = p(\alpha)\partial_i$)

➔ $\mathrm{gr}\,(\mathcal{M}) = \mathcal{A}/\mathcal{A}^{1\times 5}\left(\chi_3^2 \quad \chi_2\chi_3 - \alpha\chi_1^2 \quad \chi_2^2 \quad \chi_1\chi_3 \quad \chi_1\chi_2\right)^T$

## Janet example

**Polynomial coefficients:** $(\Sigma_2 : \;\; u_{33} = x^2 u_{11}, \;\; u_{22} = 0)$

- $R = \begin{pmatrix} \partial_3^2 - x^2\partial_1^2 & \partial_2^2 \end{pmatrix}^T$

- $\mathcal{D} = \mathbb{A}_3 := \mathbb{Q}[x^1, x^2, x^3]\langle \partial_1, \partial_2, \partial_3 \rangle$: polynomial Weyl algebra

  $\mathbb{Q}[x^1, x^2, x^3]$ with usual derivations: $\partial_i x^j = x^j \partial_i + \delta_i^j$

- $\mathcal{A} = \mathbb{Q}[x^1, x^2, x^3][\chi_1, \chi_2, \chi_3]$

➔ $\mathrm{gr}\,(\mathcal{M}) = \mathcal{A}/\mathcal{A}^{1\times 2}\begin{pmatrix} \chi_3^2 - x^2\chi_1^2 & \chi_2^2 \end{pmatrix}^T$

**Remark:** $\mathcal{D} \not\simeq \mathcal{A}$ as rings but $\mathcal{D} \simeq \mathcal{A}$ as vector spaces

## Koszul complex

Let $\mathcal{M}$ be a graded $\mathcal{A} := \mathcal{B}[\chi_1, \cdots, \chi_n]$-module ($\mathcal{B}$: a commutative ring)

**Tangent module:** $T := \mathcal{A}_1 = \mathcal{B}\chi_1 \oplus \cdots \oplus \mathcal{B}\chi_n \simeq \mathcal{B}^n$

**Exterior algebra:** $\Lambda^\bullet T := \oplus \Lambda^k T = \mathcal{B} \oplus T \oplus \Lambda^2 T \oplus \cdots \oplus \Lambda^n T$

- ➔ $\mathcal{B}$-module generated by $\chi_1, \cdots, \chi_n$ subject to $\chi_i \wedge \chi_j = -\chi_j \wedge \chi_i$ (in particular: $\chi_i \wedge \chi_i = 0$)
- ➔ free $\mathcal{B}$-module with basis $\chi_{i_1} \wedge \cdots \wedge \chi_{i_k}$, $i_1 < \cdots < i_k$

**Koszul differential:** $\partial_{k+1} : \Lambda^{k+1} T \otimes \mathcal{M} \to \Lambda^k T \otimes \mathcal{M}$ defined by

$$\partial_{k+1}\left(\chi_{i_1} \wedge \cdots \wedge \chi_{i_{k+1}} \otimes m\right) = \sum_{j=1}^{k+1} (-1)^{j+1} \chi_{i_1} \wedge \cdots \wedge \widehat{\chi_{i_j}} \wedge \cdots \wedge \chi_{i_{k+1}} \otimes \chi_{i_j} m$$

**Remark:** $\partial_k$ is $\mathcal{A}$-linear: $\partial_k(p(\mathbf{v} \otimes m)) = \partial_k(\mathbf{v} \otimes pm) = p\partial_k(\mathbf{v} \otimes m)$

## Koszul complex II

Let $\mathcal{M}$ be a graded $\mathcal{A} := \mathcal{B}[\chi_1, \cdots, \chi_n]$-module

**Prop.:** the Koszul complex $(\Lambda^\bullet T, \partial_\bullet)$ is a complex of $\mathcal{A}$-modules

**Proof:** $\partial_k \circ \partial_{k+1}\left(\chi_{i_1} \wedge \cdots \wedge \chi_{i_{k+1}} \otimes m\right)$ is equal to

$$\partial_k\left(\sum_{j=1}^{k+1}(-1)^{j+1}\chi_{i_1} \wedge \cdots \wedge \widehat{\chi_{i_j}} \wedge \cdots \wedge \chi_{i_{k+1}} \otimes \chi_{i_j} m\right)$$

$$= \sum_{j=1}^{k+1}\sum_{l=1}^{j-1}(-1)^{j+l}\chi_{i_1} \wedge \cdots \wedge \widehat{\chi_{i_l}} \wedge \cdots \wedge \widehat{\chi_{i_j}} \wedge \cdots \wedge \chi_{i_{k+1}} \otimes \chi_{i_l}\chi_{i_j} m$$

$$+ \sum_{j=1}^{k+1}\sum_{l=j+1}^{k+1}(-1)^{j+l+1}\chi_{i_1} \wedge \cdots \wedge \widehat{\chi_{i_j}} \wedge \cdots \wedge \widehat{\chi_{i_l}} \wedge \cdots \wedge \chi_{i_{k+1}} \otimes \chi_{i_l}\chi_{i_j} m$$

$$= 0$$

## 2-acyclicity and involution

Let $(\Sigma_d : Ru = 0)$ and $\mathrm{gr}\,(\mathcal{M})$ be the symbol module of $\Sigma_d$

$(R \in \mathcal{D}^{q\times p}$, $\mathcal{D} = \mathcal{B}\langle\partial_1, \cdots, \partial_n\rangle$, $\mathcal{A} := \mathrm{gr}(\mathcal{D}) = \mathcal{B}[\chi_1, \cdots, \chi_n])$

**Koszul homology of $\mathrm{gr}\,(\mathcal{M})$:** the Koszul complex (of $\mathcal{A}$-modules) is

$$0 \to \Lambda^n T \otimes \mathrm{gr}\,(\mathcal{M}) \xrightarrow{\partial_n} \Lambda^{n-1} T \otimes \mathrm{gr}\,(\mathcal{M}) \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} T \otimes \mathrm{gr}\,(\mathcal{M}) \xrightarrow{\partial_1} \mathrm{gr}\,(\mathcal{M}) \to 0$$

The **Koszul homology** of $\mathrm{gr}\,(\mathcal{M})$ is the $\mathcal{A}$-module ($\partial_{n+1} := 0$ and $\partial_0 := 0$)

$$H_\bullet(\mathrm{gr}\,(\mathcal{M})) := \bigoplus_{k=0}^{n} H_k(\mathrm{gr}\,(\mathcal{M})) := \bigoplus_{k=0}^{n} \frac{\ker(\partial_k)}{\mathrm{im}(\partial_{k+1})}$$

**Definition.** The symbol module $\mathrm{gr}\,(\mathcal{M})$ of $\Sigma_d$ is said to be

- ➔ 2-acyclic if $H_1(\mathrm{gr}\,(\mathcal{M})) = H_2(\mathrm{gr}\,(\mathcal{M})) = 0$
- ➔ involutive if $H_1(\mathrm{gr}\,(\mathcal{M})) = \cdots = H_n(\mathrm{gr}\,(\mathcal{M})) = 0$

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## Computation of homology modules

Let $\mathcal{M}_i = \mathcal{A}^{1\times p_i} / \left(\mathcal{A}^{1\times q_i} R_i\right)$ and $\mathcal{M}_2 \xrightarrow{f} \mathcal{M}_1 \xrightarrow{g} \mathcal{M}_0$ be a complex of $\mathcal{A}$-modules

**Fact:** $\exists$ $P_i$'s and $Q_i$'s such that the following diagram commutes

$$\begin{array}{ccccccc}
\mathcal{A}^{1\times q_2} & \xrightarrow{.R_2} & \mathcal{A}^{1\times p_2} & \longrightarrow & \mathcal{M}_2 & \longrightarrow & 0 \\
\downarrow{\scriptstyle .Q_2} & & \downarrow{\scriptstyle .P_2} & & \downarrow{\scriptstyle f} & & \\
\mathcal{A}^{1\times q_1} & \xrightarrow{.R_1} & \mathcal{A}^{1\times p_1} & \longrightarrow & \mathcal{M}_1 & \longrightarrow & 0 \\
\downarrow{\scriptstyle .Q_1} & & \downarrow{\scriptstyle .P_1} & & \downarrow{\scriptstyle g} & & \\
\mathcal{A}^{1\times q_0} & \xrightarrow{.R_0} & \mathcal{A}^{1\times p_0} & \longrightarrow & \mathcal{M}_0 & \longrightarrow & 0
\end{array}$$

- generators of $\mathcal{M}_i$ are mapped to lin. comb. of generators of $\mathcal{M}_{i-1}$ ➔ $\exists P_i$
- generating relations of $\mathcal{M}_i$ are mapped to 0 in $\mathcal{M}_{i-1}$ ➔ $\exists Q_i$ st $R_i P_i = Q_i R_{i-1}$

**Presentation of $H$:** $\exists S', S'' :$ $\ker \begin{pmatrix} .P_1 \\ .R_0 \end{pmatrix} = \mathcal{A}^{1\times s'} \begin{pmatrix} S' & -S'' \end{pmatrix}$ and

$$\ker(g)/\operatorname{im}(f) = \mathcal{A}^{1\times s'} S' / \left(\mathcal{A}^{1\times (p_2+q_1)} \begin{pmatrix} P_2 \\ R_1 \end{pmatrix}\right)$$

**Implementation:** if $\mathcal{A}$ is an Ore algebra, `OreModules` computes $S'$ and $S''$

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\downarrow {\scriptstyle .Q_1} & & \downarrow {\scriptstyle .P_1} & & \downarrow {\scriptstyle g} & & \\
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## How to prove that homology vanishes?

Let $\mathcal{M}_2 \xrightarrow{f} \mathcal{M}_1 \xrightarrow{g} \mathcal{M}_0$ be a complex of $\mathcal{A}$-modules

**Homology module:** $H(\mathcal{M}_2 \to \mathcal{M}_1 \to \mathcal{M}_0) = \mathcal{A}^{1\times s'} S' / \mathcal{A}^{1\times s} S$

- ➔ lines of $S'$ represent generators of $\ker(g)$
- ➔ lines of $S$ represent generators of $\operatorname{im}(f)$

**Acyclicity criterion:** the following are equivalent

- ➔ $H\left(\mathcal{M}_2 \xrightarrow{f} \mathcal{M}_1 \xrightarrow{g} \mathcal{M}_0\right) = 0$
- ➔ $\mathcal{A}^{1\times s'} S' \subseteq \mathcal{A}^{1\times s} S$
- ➔ there exists $T \in \mathcal{A}^{s'\times s}$ such that $S' = TS$

**Implementation:** if $\mathcal{A}$ is an Ore algebra, `OreModules` computes $T$ (if it exists)

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## Computation of Koszul homology

$\mathrm{gr}\,(\mathcal{M}) := \mathcal{A}^{1\times p}/\mathcal{A}^{1\times q}\sigma_d(R)$: symbol module of $(\Sigma_d : Ru = 0)$

**Presentation of $\Lambda^k T \otimes \mathrm{gr}\,(\mathcal{M})$:** $\mathcal{A}^{1\times r_k p}/\mathcal{A}^{1\times r_k q}\sigma_d(R)$, $r_k := \dim_{\mathcal{B}} \left(\Lambda^k T\right)$

**Matrix representation of $\partial$:**

$$\begin{array}{ccccccc}
\mathcal{A}^{1\times r_{k+1} q} & \xrightarrow{.I_{r_{k+1}}\otimes\sigma_d(R)} & \mathcal{A}^{1\times r_{k+1} p} & \longrightarrow & \Lambda^{k+1} T \otimes \mathrm{gr}\,(\mathcal{M}) & \longrightarrow & 0 \\
\downarrow {\scriptstyle .P_{k+1}\otimes I_q} & & \downarrow {\scriptstyle .P_{k+1}\otimes I_p} & & \downarrow {\scriptstyle \partial_{k+1}} & & \\
\mathcal{A}^{1\times r_k q} & \xrightarrow{.I_{r_k}\otimes\sigma_d(R)} & \mathcal{A}^{1\times r_k p} & \longrightarrow & \Lambda^{k} T \otimes \mathrm{gr}\,(\mathcal{M}) & \longrightarrow & 0 \\
\downarrow {\scriptstyle .P_{k}\otimes I_q} & & \downarrow {\scriptstyle .P_{k}\otimes I_p} & & \downarrow {\scriptstyle \partial_{k}} & & \\
\mathcal{A}^{1\times r_{k-1} q} & \xrightarrow{.I_{r_{k-1}}\otimes\sigma_d(R)} & \mathcal{A}^{1\times r_{k-1} p} & \longrightarrow & \Lambda^{k-1} T \otimes \mathrm{gr}\,(\mathcal{M}) & \longrightarrow & 0
\end{array}$$

➔ $P_k$: analogous to grad, rot, div

**Consequence:** 2-acyclicity may be proven using `OreMorphisms` and `OreModules`

**Remark:** involution may be proven analogously

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\end{array}
$$

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**Theorem (C., Cluzeau, Quadrat; 2022).**

Let $\Sigma_d$ be a linear system of PDEs and let $\operatorname{gr}(\mathcal{M})$ be the symbol module of $\Sigma_d$.
Let $S, S'$ be two matrices such that

$$H_2(\operatorname{gr}(\mathcal{M})) \;=\; \mathcal{A}^{1\times s'} S' / \left(\mathcal{A}^{1\times s} S\right)$$

Then, $\operatorname{gr}(\mathcal{M})$ is 2-acyclic iff the top degree part of $S'$ is a left-multiple of $S$.

# III. Examples

## Example 1

$(\Sigma_2 : \ u_{33} = u_{13}, \ u_{23} = u_{13}, \ u_{12} = u_{11}), \quad \mathcal{A} = \mathbb{Q}[\chi_1, \chi_2, \chi_3]$

**Symbol module:** $\mathrm{gr}(\mathcal{M}) = \mathcal{A}/\mathcal{A}^{1\times 3} \begin{pmatrix} \chi_3^2 - \chi_1\chi_3 & \chi_2\chi_3 - \chi_1\chi_3 & \chi_1\chi_2 - \chi_1^2 \end{pmatrix}^T$

**Koszul complex (degree 2):** $\Lambda^3 T \otimes \mathrm{gr}(\mathcal{M}) \xrightarrow{\partial_3} \Lambda^2 T \otimes \mathrm{gr}(\mathcal{M}) \xrightarrow{\partial_2} T \otimes \mathrm{gr}(\mathcal{M})$

- ➔ $\partial_3(\chi_1 \wedge \chi_2 \wedge \chi_3 \otimes m) = \chi_2 \wedge \chi_3 \otimes \chi_1 m - \chi_1 \wedge \chi_3 \otimes \chi_2 m + \chi_1 \wedge \chi_3 \otimes \chi_3 m$
- ➔ $\partial_2(\chi_i \wedge \chi_j \otimes m) = \chi_j \otimes \chi_i m - \chi_i \otimes \chi_j m.$

**Ex. of 2-cycle:** $\mathbf{c} := \chi_3 \wedge \chi_1 \otimes (\chi_1\chi_2 - \chi_2^2)$

- ➔ $\partial_2(\mathbf{c}) = \chi_1 \otimes \underbrace{(\chi_1\chi_2\chi_3 - \chi_2^2\chi_3)}_{-\chi_2(\chi_2\chi_3 - \chi_1\chi_3)} + \chi_3 \otimes \underbrace{(-\chi_1^2\chi_2 + \chi_1\chi_2^2)}_{\chi_2(\chi_1\chi_2 - \chi_1^2)} = 0$

**Ex. of 2-boundary:** $\mathbf{c} = \partial_3(\mathbf{v})$, $\mathbf{v} := \chi_1 \wedge \chi_2 \wedge \chi_3 \otimes (\chi_1 - \chi_2)$; indeed

$$\partial_3(\mathbf{v}) = \chi_2 \wedge \chi_3 \otimes \underbrace{(\chi_1^2 - \chi_1\chi_2)}_{0} + \underbrace{\chi_3 \wedge \chi_1 \otimes (\chi_1\chi_2 - \chi_2^2)}_{\mathbf{c}} + \chi_1 \wedge \chi_2 \otimes \underbrace{(\chi_1\chi_3 - \chi_2\chi_3)}_{0} = \mathbf{c}$$

## Example 1 (continued)

$$\mathrm{gr}\,(\mathcal{M}) = \mathcal{A}/\mathcal{A}^{1\times 3}\left(\chi_3^2 - \chi_1\chi_3 \quad \chi_2\chi_3 - \chi_1\chi_3 \quad \chi_1\chi_2 - \chi_1^2\right)^T$$

**Koszul complex (degree 2):** $\Lambda^3 T \otimes \mathrm{gr}\,(\mathcal{M}) \xrightarrow{\partial_3} \Lambda^2 T \otimes \mathrm{gr}\,(\mathcal{M}) \xrightarrow{\partial_2} T \otimes \mathrm{gr}\,(\mathcal{M})$

- ➔ the $\mathcal{A}$-module $\Lambda^2 T \otimes \mathrm{gr}\,(\mathcal{M})$ is generated by $\left\{\chi_2 \wedge \chi_3 \otimes 1, \chi_3 \wedge \chi_1 \otimes 1, \chi_1 \wedge \chi_2 \otimes 1\right\}$

**$H_2$ Koszul homology:** $H_2(\mathrm{gr}\,(\mathcal{M})) = \mathcal{A}^{1\times 7} S' / \left(\mathcal{A}^{1\times 10} S\right)$

$$S' = \begin{pmatrix} \chi_3 & \chi_3 & \chi_3 \\ \chi_1 & \chi_2 & \chi_3 \\ 0 & \chi_1 - \chi_2 & 0 \\ 0 & \chi_2\chi_3 - \chi_3^2 & 0 \\ 0 & 0 & \chi_2\chi_3 - \chi_3^2 \\ 0 & 0 & \chi_1\chi_3 - \chi_3^2 \\ 0 & 0 & \chi_1^2 - \chi_1\chi_2 \end{pmatrix} \qquad S = \begin{pmatrix} \chi_1 & \chi_2 & \chi_3 \\ \chi_3^2 - \chi_1\chi_3 & 0 & 0 \\ \chi_2\chi_3 - \chi_1\chi_3 & 0 & 0 \\ \chi_1\chi_2 - \chi_1^2 & 0 & 0 \\ 0 & \chi_3^2 - \chi_1\chi_3 & 0 \\ 0 & \chi_2\chi_3 - \chi_1\chi_3 & 0 \\ 0 & \chi_1\chi_2 - \chi_1^2 & 0 \\ 0 & 0 & \chi_3^2 - \chi_1\chi_3 \\ 0 & 0 & \chi_2\chi_3 - \chi_1\chi_3 \\ 0 & 0 & \chi_1\chi_2 - \chi_1^2 \end{pmatrix}$$

**Remarks.**

- ➔ columns of $S$ and $S'$ $\leftrightarrow$ $\left\{\mathbf{u} := \chi_2 \wedge \chi_3 \otimes 1,\ \mathbf{v} := \chi_3 \wedge \chi_1 \otimes 1,\ \mathbf{w} := \chi_1 \wedge \chi_2 \otimes 1\right\}$
- ➔ $\chi_3\,(\mathbf{u} + \mathbf{v} + \mathbf{w})$ and $(\chi_1 - \chi_2)\mathbf{v}$ are $\neq 0$ in $H_2(\mathrm{gr}\,(\mathcal{M}))$, but have degrees $1 \neq \deg(S')$

**2-acyclicity test:** **success**

## Example 2

$(\Sigma_2 : \ u_{33} = u_{22} = u_{13} = u_{12} = 0, \ u_{23} = \alpha u_{11}), \quad \mathcal{A} = \mathbb{Q}[\alpha][\chi_1, \chi_2, \chi_3]$

**Symbol module:** $\mathrm{gr}\,(\mathcal{M})_\alpha = \mathcal{A}/\mathcal{A}^{1\times 5}\begin{pmatrix}\chi_3^2 & \chi_2\chi_3 - \alpha\chi_1^2 & \chi_2^2 & \chi_1\chi_3 & \chi_1\chi_2\end{pmatrix}^T$

**Koszul complex (degree 2):** $\Lambda^3 T \otimes \mathrm{gr}\,(\mathcal{M})_\alpha \xrightarrow{\partial_3} \Lambda^2 T \otimes \mathrm{gr}\,(\mathcal{M})_\alpha \xrightarrow{\partial_2} T \otimes \mathrm{gr}\,(\mathcal{M})_\alpha$

➔ $\partial_3(\chi_1 \wedge \chi_2 \wedge \chi_3 \otimes m) = \chi_2 \wedge \chi_3 \otimes \chi_1 m - \chi_1 \wedge \chi_3 \otimes \chi_2 m + \chi_1 \wedge \chi_3 \otimes \chi_3 m$

➔ $\partial_2(\chi_i \wedge \chi_j \otimes m) = \chi_j \otimes \chi_i m - \chi_i \otimes \chi_j m.$

**Ex. of 2-cycle:** $\mathbf{c} := \chi_3 \wedge \chi_1 \otimes \alpha\chi_1^2$

➔ $\partial_2(\mathbf{c}) = \chi_1 \otimes \underbrace{\alpha\chi_1^2\chi_3}_{\alpha\chi_1(\chi_1\chi_3)} + \chi_3 \otimes (-\alpha\chi_1^3) = \chi_3 \otimes \Big(\big(\underbrace{\chi_1\chi_2\chi_3 - \alpha\chi_1^3}_{\chi_1(\chi_2\chi_3 - \alpha\chi_1^2)}\big) - \underbrace{\chi_1\chi_2\chi_3}_{\chi_3(\chi_1\chi_2)}\Big) = 0$

**Ex. of 2-boundary:** $\mathbf{c} = \partial_3(\mathbf{v})$, $\mathbf{v} := \chi_1 \wedge \chi_2 \wedge \chi_3 \otimes \chi_3$; indeed

➔ $\partial_3(\mathbf{v}) = \chi_2 \wedge \chi_3 \otimes \chi_1\chi_3 + \chi_3 \wedge \chi_1 \otimes \chi_2\chi_3 + \chi_1 \wedge \chi_2 \otimes \chi_3^2 = \mathbf{c}$

## Example 2 (continued)

$$\mathrm{gr}\,(\mathcal{M})_\alpha = \mathcal{A}/\mathcal{A}^{1\times 5}\left(\chi_3^2 \quad \chi_2\chi_3 - \alpha\chi_1^2 \quad \chi_2^2 \quad \chi_1\chi_3 \quad \chi_1\chi_2\right)^T$$

**Koszul complex (degree 2):** $\Lambda^3 T \otimes \mathrm{gr}\,(\mathcal{M})_\alpha \xrightarrow{\partial_3} \Lambda^2 T \otimes \mathrm{gr}\,(\mathcal{M})_\alpha \xrightarrow{\partial_1} T \otimes \mathrm{gr}\,(\mathcal{M})_\alpha$

$H_2$ **Koszul homology:** $H_2\left(\mathrm{gr}\,(\mathcal{M})_\alpha\right) = \mathcal{A}^{1\times 11} S' / \left(\mathcal{A}^{1\times 16} S\right)$

$$S' = \begin{pmatrix} \chi_2 & 0 & \alpha\chi_1 \\ \chi_3 & \alpha\chi_1 & 0 \\ \chi_1 & 0 & 0 \\ \chi_3^2 & 0 & 0 \\ \chi_2\chi_3 & 0 & 0 \\ \chi_2^2 & 0 & 0 \\ 0 & \chi_3 & 0 \\ 0 & \chi_2 & \chi_3 \\ 0 & 0 & \chi_2 \\ 0 & 0 & \chi_3^2 \\ 0 & 0 & \chi_1\chi_3 \end{pmatrix} \qquad S = \begin{pmatrix} \chi_1 & \chi_2 & \chi_3 \\ \chi_3^2 & 0 & 0 \\ \chi_2\chi_3 - \alpha\chi_1^2 & 0 & 0 \\ \chi_2^2 & 0 & 0 \\ \chi_1\chi_3 & 0 & 0 \\ \chi_1\chi_2 & 0 & 0 \\ 0 & \chi_3^2 & 0 \\ 0 & \chi_2\chi_3 - \alpha\chi_1^2 & 0 \\ 0 & \chi_2^2 & 0 \\ 0 & \chi_1\chi_3 & 0 \\ 0 & \chi_1\chi_2 & 0 \\ 0 & 0 & \chi_3^2 \\ 0 & 0 & \chi_2\chi_3 - \alpha\chi_1^2 \\ 0 & 0 & \chi_2^2 \\ 0 & 0 & \chi_1\chi_3 \\ 0 & 0 & \chi_1\chi_2 \end{pmatrix}$$

**2-acyclicity test:** **success**.

## Example 2 (continued)

$$\mathrm{gr}\,(\mathcal{M})_\alpha = \mathcal{A}/\mathcal{A}^{1\times 5}\begin{pmatrix}\chi_3^2 & \chi_2\chi_3-\alpha\chi_1^2 & \chi_2^2 & \chi_1\chi_3 & \chi_1\chi_2\end{pmatrix}^T$$

**Koszul complex (degree 2):** $\Lambda^3 T\otimes \mathrm{gr}\,(\mathcal{M})_\alpha \xrightarrow{\partial_3} \Lambda^2 T\otimes \mathrm{gr}\,(\mathcal{M})_\alpha \xrightarrow{\partial_1} T\otimes \mathrm{gr}\,(\mathcal{M})_\alpha$

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**2-acyclicity test:** **success**. Moreover, $\mathrm{gr}\,(\mathcal{M})_\alpha$ is involutive iff $\alpha \neq 0$

## Example 3 (Janet example)

$(\Sigma_2: \; u_{33} = x^2 u_{11}, \; u_{22} = 0), \quad \mathcal{A} := \mathbb{Q}[x^1, x^2, x^3][\chi_1, \chi_2, \chi_3]$

**Symbol module:** $\mathrm{gr}(\mathcal{M}) = \mathcal{A}/\mathcal{A}^{1\times 2} \begin{pmatrix} \chi_3^2 - x^2\chi_1^2 & \chi_2^2 \end{pmatrix}^T$

**Notation:** $\mathbf{u} := \chi_2 \wedge \chi_3, \; \mathbf{v} := \chi_3 \wedge \chi_1, \; \mathbf{w} := \chi_1 \wedge \chi_2$

➔ generate $\Lambda^2 \otimes \mathrm{gr}(\mathcal{M})$ as an $\mathcal{A}$-module

**Homology defect:** $\mathbf{c} := \mathbf{u} \otimes \chi_2\chi_3 + \mathbf{w} \otimes x^2\chi_1\chi_2 = \chi_2 \wedge \chi_3 \otimes \chi_2\chi_3 + \chi_1 \wedge \chi_2 \otimes x^2\chi_1\chi_2$

➔ $\partial_2(\mathbf{c}) = \chi_1 \otimes \Big(\underbrace{-x^2\chi_1\chi_2^2}_{-x^2\chi_1(\chi_2^2)}\Big) + \chi_2 \otimes \Big(\underbrace{-\chi_2\chi_3^2 + x^2\chi_1^2\chi_2}_{-\chi_2(\chi_3^2 - x^2\chi_1^2)}\Big) + \chi_3 \otimes \Big(\underbrace{\chi_2^2\chi_3}_{\chi_3(\chi_2^2)}\Big) = 0$

Moreover, $\partial_3\big(\mathbf{x} := \chi_1 \wedge \chi_2 \wedge \chi_3 \otimes (P_1\chi_1 + P_2\chi_2 + P_3\chi_3)\big) = \lambda\mathbf{u} + \mu\mathbf{v} + \nu\mathbf{w}$, where

$$\lambda \;=\; P_1\chi_1^2 + P_2\chi_1\chi_2 + P_3\chi_1\chi_3, \quad \mu \;=\; \cdots, \quad \nu = \cdots$$

Hence, $\partial_3(\mathbf{x}) = \mathbf{c}$ ➔ $P_1\chi_1^2 + P_2\chi_1\chi_2 + P_3\chi_1\chi_3 = \chi_2\chi_3$: IMPOSSIBLE!

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Moreover, $\partial_3\big(\mathbf{x} := \chi_1 \wedge \chi_2 \wedge \chi_3 \otimes (P_1\chi_1 + P_2\chi_2 + P_3\chi_3)\big) = \lambda\mathbf{u} + \mu\mathbf{v} + \nu\mathbf{w}$, where

$$\lambda = P_1\chi_1^2 + P_2\chi_1\chi_2 + P_3\chi_1\chi_3, \quad \mu = \cdots, \quad \nu = \cdots$$

Hence, $\partial_3(\mathbf{x}) = \mathbf{c}$ ➔ $P_1\chi_1^2 + P_2\chi_1\chi_2 + P_3\chi_1\chi_3 = \chi_2\chi_3$: **IMPOSSIBLE!**

## Example 3 (continued)

$$\mathrm{gr}\,(\mathcal{M}) \;=\; \mathcal{A}/\mathcal{A}^{1\times 2}\begin{pmatrix}\chi_3^2 - x^2\chi_1^2 & \chi_2^2\end{pmatrix}^T$$

**Koszul complex (degree 2):** $\Lambda^3 T \otimes \mathrm{gr}\,(\mathcal{M}) \xrightarrow{\partial_3} \Lambda^2 T \otimes \mathrm{gr}\,(\mathcal{M}) \xrightarrow{\partial_1} T \otimes \mathrm{gr}\,(\mathcal{M})$

*$H_2$* **Koszul homology:** $H_2(\mathrm{gr}\,(\mathcal{M})) = \mathcal{A}^{1\times 8} S' / \left(\mathcal{A}^{1\times 7} S\right)$

$$S' \;=\; \begin{pmatrix} \chi_1 & \chi_2 & \chi_3 \\ \chi_2^2 & 0 & 0 \\ \chi_2\chi_3 & 0 & x^2\chi_1\chi_2 \\ \chi_3^2 & x^2\chi_1\chi_2 & x^2\chi_1\chi_3 \\ 0 & \chi_2^2 & 0 \\ 0 & x^2\chi_1^2 - \chi_3^2 & 0 \\ 0 & 0 & \chi_2^2 \\ 0 & 0 & x^2\chi_1^2 - \chi_3^2 \end{pmatrix} \qquad S \;=\; \begin{pmatrix} \chi_1 & \chi_2 & \chi_3 \\ \chi_3^2 - x^2\chi_1^2 & 0 & 0 \\ \chi_2^2 & 0 & 0 \\ 0 & \chi_3^2 - x^2\chi_1^2 & 0 \\ 0 & \chi_2^2 & 0 \\ 0 & 0 & \chi_3^2 - x^2\chi_1^2 \\ 0 & 0 & \chi_2^2 \end{pmatrix}$$

**2-acyclicity test:** **fail**

# IV. CONCLUSION AND PERSPECTIVES

## Conclusion and perspectives

### Summary of presented results

- ➔ new criterion for proving 2-ayclicity of linear PDEs systems
- ➔ effective test using `OreMorphisms` and `OreModules` packages

**Remark.** Involution can be checked with `OreMorphism` and `OreModules`

### Further works

- ➔ go further into the effective approach to Spencer cohomology
  (Koszul-Tate theory, Spencer sequences)
- ➔ applications to physics and control theory
  (elasticity theory, minimal realization, $\cdots$)
- ➔ investigate rewriting systems over jet bundles
  (differential elimination, homotopy of rewriting systems)

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**THANK YOU FOR LISTENING!**