Actions of Frobenii for moduli stacks of principal bundles

Frank Neumann

Dipartimento di Matematica ”Felice Casorati”
Università di Pavia

Champs algébriques et catégories dérivées
XLIM Limoges
December 6, 2024

I. Moduli stacks of vector bundles

Definition

A **vector bundle** (over a scheme X) over a field k is a scheme $\mathcal{E}$ together with a map $\pi : \mathcal{E} \to X$ of schemes such that π is **locally trivial** in the (Zariski) topology, i.e. there is a (Zariski) open covering $\{U_i\}_{i \in I}$ of X and isomorphisms

$$\varphi_i : \pi^{-1}(U_i) \xrightarrow{\cong} U_i \times \mathbb{A}^n_k$$

such that for every pair $i, j \in I$ there is a morphism, a **transition function**

$$\varphi_{ij} : U_i \cap U_j \to GL_n(k)$$

such that $\varphi_i \varphi_j^{-1}(x, v) = (x, \varphi_{ij}(x)v)$ for all $x \in U_i \cap U_j$ and $v \in \mathbb{A}^n_k$.

I. Moduli stacks of vector bundles

Definition

A **vector bundle** (over a scheme X) over a field k is a scheme $\mathcal{E}$ together with a map $\pi : \mathcal{E} \to X$ of schemes such that π is **locally trivial** in the (Zariski) topology, i.e. there is a (Zariski) open covering $\{U_i\}_{i \in I}$ of X and isomorphisms

$$\varphi_i : \pi^{-1}(U_i) \stackrel{\cong}{\to} U_i \times \mathbb{A}_k^n$$

such that for every pair $i, j \in I$ there is a morphism, a **transition function**

$$\varphi_{ij} : U_i \cap U_j \to GL_n(k)$$

such that $\varphi_i \varphi_j^{-1}(x, v) = (x, \varphi_{ij}(x)v)$ for all $x \in U_i \cap U_j$ and $v \in \mathbb{A}_k^n$.
$n = \mathrm{rk}(\mathcal{E})$ is the **rank** of $\mathcal{E}$.

I. Moduli stacks of vector bundles

Definition

A **vector bundle** (over a scheme X) over a field k is a scheme $\mathcal{E}$ together with a map $\pi : \mathcal{E} \to X$ of schemes such that π is **locally trivial** in the (Zariski) topology, i.e. there is a (Zariski) open covering $\{U_i\}_{i \in I}$ of X and isomorphisms

$$\varphi_i : \pi^{-1}(U_i) \overset{\cong}{\to} U_i \times \mathbb{A}^n_k$$

such that for every pair $i, j \in I$ there is a morphism, a **transition function**

$$\varphi_{ij} : U_i \cap U_j \to GL_n(k)$$

such that $\varphi_i \varphi_j^{-1}(x, v) = (x, \varphi_{ij}(x)v)$ for all $x \in U_i \cap U_j$ and $v \in \mathbb{A}^n_k$.
$n = \mathrm{rk}(\mathcal{E})$ is the **rank** of $\mathcal{E}$. If $\mathrm{rk}(\mathcal{E}) = 1$, then $\mathcal{E}$ is a **line bundle.**

I. Moduli stacks of vector bundles

Definition

Let $\pi : \mathcal{E} \to X$ be a vector bundle of rank n with trivializations $(U_i, \varphi_i, \varphi_{ij})$ and $\pi' : \mathcal{E}' \to X$ be a vector bundle of rank n' with trivializations $(U_i', \varphi_i', \varphi_{ij}')$. A **morphism of vector bundles** $f : \mathcal{E} \to \mathcal{E}'$ is given by a commutative diagram

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\ f\ } & \mathcal{E}' \\ & {\scriptstyle \pi}\searrow \quad \swarrow{\scriptstyle \pi'} & \\ & X & \end{array}$$

such that for every pair $i, j \in I$ there is a morphism

$$f_{ij} : U_i' \cap U_j \to Mat_{n \times n'}(k)$$

such that $\varphi_i' f \varphi_j^{-1}(x, v) = (x, f_{ij}(x)v)$ for all $x \in U_i' \cap U_j$ and $v \in \mathbb{A}_k^n$.

I. Moduli stacks of vector bundles

Definition

A vector bundle is **trivial** if it is isomorphic to $pr_1 : X \times \mathbb{A}_k^n \to X$.

I. Moduli stacks of vector bundles

Definition

A vector bundle is **trivial** if it is isomorphic to $pr_1 : X \times \mathbb{A}_k^n \to X$.

Definition

The **degree** $\deg(\mathcal{E})$ of a vector bundle $\mathcal{E}$ over an algebraic curve X is the degree of divisor of the determinant line bundle $\det(\mathcal{E}) = \Lambda^{\mathrm{rk}(\mathcal{E})}(\mathcal{E})$, i.e. $\deg(\mathcal{E}) = \dim H^0(X, \mathcal{E}) - \dim H^1(X, \mathcal{E}) - \mathrm{rk}(\mathcal{E})(1 - g)$.

I. Moduli stacks of vector bundles

Definition

A vector bundle is **trivial** if it is isomorphic to $pr_1 : X \times \mathbb{A}^n_k \to X$.

Definition

The **degree** $\deg(\mathcal{E})$ of a vector bundle $\mathcal{E}$ over an algebraic curve X is the degree of divisor of the determinant line bundle $\det(\mathcal{E}) = \Lambda^{\mathrm{rk}(\mathcal{E})}(\mathcal{E})$, i.e. $\deg(\mathcal{E}) = \dim H^0(X, \mathcal{E}) - \dim H^1(X, \mathcal{E}) - \mathrm{rk}(\mathcal{E})(1-g)$.

Remark. Have also more general bundles like **principal G-bundles**, i.e. a fiber bundle where the total space $\mathcal{E}$ has an action of an algebraic group G, which for $G = GL_n(k)$ corresponds to vector bundles.

I. Moduli stacks of vector bundles

Definition

Let X be a scheme over a field k and G an affine algebraic group over k. A **G-fibration** over X is given by a scheme $\mathcal{P}$, an action $\rho : \mathcal{P} \times G \to \mathcal{P}$ and a G-equivariant morphism $\pi : \mathcal{P} \to X$. A **morphism** between two G-fibrations $\pi : \mathcal{P} \to X$ and $\pi' : \mathcal{P}' \to X$ is given by a morphism $f : \mathcal{P} \to \mathcal{P}'$ such that $\pi = \pi' \circ f$, i.e. by a commutative diagram

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{f} & \mathcal{P}' \\ & {\scriptstyle \pi} \searrow \quad \swarrow {\scriptstyle \pi'} & \\ & X & \end{array}$$

A G-fibration is called **trivial** if it is isomorphic to the G-fibration $pr_1 : X \times G \to X$, where the action is given by

$$\rho : (X \times G) \times G \to X \times G, \; \rho((x, g), g') = (x, gg').$$

I. Moduli stacks of vector bundles

A principal G-bundle is now simply a locally trivial G-fibration. But it is important to specify local triviality with respect to a given topology:

Definition

Let X be a scheme over a field k and G an affine algebraic group over k. A **principal G-bundle** in the **Zariski** (resp. **étale**...) topology is a G-fibration $\mathcal{P}$ which is locally trivial in the Zariski (resp. étale ...) topology. This means that for any point $x \in X$ there is a neighborhood U of x such that $\mathcal{P}|_U$ is trivial in the Zariski topology, resp. there is an étale ... covering $U' \xrightarrow{\varphi} U$ such that the fibre product

$$\varphi^*(\mathcal{P}|_U) \cong U' \times_U \mathcal{P}|_U$$

is trivial.

I. Moduli stacks of vector bundles

Aims:

I. Moduli stacks of vector bundles

Aims:

- We want to “classify” vector bundles $\mathcal{E}$ (resp. principal G-bundles $\mathcal{P}$) on a given smooth projective algebraic curve X over a finite field $\mathbb{F}_q$ up to their symmetries, i.e. bundle isomorphism.

I. Moduli stacks of vector bundles

Aims:

- We want to "classify" vector bundles $\mathcal{E}$ (resp. principal G-bundles $\mathcal{P}$) on a given smooth projective algebraic curve X over a finite field $\mathbb{F}_q$ up to their symmetries, i.e. bundle isomorphism.
- We want to "count" the number of isomorphism classes of these vector bundles $\mathcal{E}$ (resp. principal G-bundles $\mathcal{P}$), i.e. need to determine the number of $\mathbb{F}_q$-rational points of some moduli "space", whose points corresponds to the isomorphism classes of the vector bundles.

I. Moduli stacks of vector bundles

Aims:

- We want to "classify" vector bundles $\mathcal{E}$ (resp. principal G-bundles $\mathcal{P}$) on a given smooth projective algebraic curve X over a finite field $\mathbb{F}_q$ up to their symmetries, i.e. bundle isomorphism.
- We want to "count" the number of isomorphism classes of these vector bundles $\mathcal{E}$ (resp. principal G-bundles $\mathcal{P}$), i.e. need to determine the number of $\mathbb{F}_q$-rational points of some moduli "space", whose points corresponds to the isomorphism classes of the vector bundles.

Ingredients:

I. Moduli stacks of vector bundles

Aims:

- We want to "classify" vector bundles $\mathcal{E}$ (resp. principal G-bundles $\mathcal{P}$) on a given smooth projective algebraic curve X over a finite field $\mathbb{F}_q$ up to their symmetries, i.e. bundle isomorphism.
- We want to "count" the number of isomorphism classes of these vector bundles $\mathcal{E}$ (resp. principal G-bundles $\mathcal{P}$), i.e. need to determine the number of $\mathbb{F}_q$-rational points of some moduli "space", whose points corresponds to the isomorphism classes of the vector bundles.

Ingredients:

- Need to calculate the l-adic cohomology of this moduli "space" of vector bundles (resp. principal G-bundles $\mathcal{P}$) on X and use Lefschetz type trace formula to count isomorphism classes via counting points of the associated moduli "space"!

I. Moduli stacks of vector bundles

Moduli Problem [Philosophy]

- **Question:** How to classify geometric objects (e.g. differentiable manifolds, algebraic varieties, schemes, vector bundles, principal G-bundles etc.) up to their symmetries (i.e. isomorphisms)?

I. Moduli stacks of vector bundles

Moduli Problem [Philosophy]

- **Question:** How to classify geometric objects (e.g. differentiable manifolds, algebraic varieties, schemes, vector bundles, principal G-bundles etc.) up to their symmetries (i.e. isomorphisms)?
- **Strategy:**

I. Moduli stacks of vector bundles

Moduli Problem [Philosophy]

- **Question:** How to classify geometric objects (e.g. differentiable manifolds, algebraic varieties, schemes, vector bundles, principal G-bundles etc.) up to their symmetries (i.e. isomorphisms)?

- **Strategy:**
  - Construct a geometric object (e.g. topological space, differentiable manifold, fibrations, foliations, algebraic variety, scheme etc.) such that its set of points corresponds bijectively to the set of isomorphism classes of the geometric objects we want to classify.

I. Moduli stacks of vector bundles

Moduli Problem [Philosophy]

- **Question:** How to classify geometric objects (e.g. differentiable manifolds, algebraic varieties, schemes, vector bundles, principal G-bundles etc.) up to their symmetries (i.e. isomorphisms)?

- **Strategy:**
  - Construct a geometric object (e.g. topological space, differentiable manifold, fibrations, foliations, algebraic variety, scheme etc.) such that its set of points corresponds bijectively to the set of isomorphism classes of the geometric objects we want to classify.
  - Construct a universal object inside the class of geometric objects we want to classify, such that all other geometric objects inside the class can be obtained from this universal object in a systematic manner.

I. Moduli stacks of vector bundles

Moduli Problem [Mathematics]

A Case Study: Moduli of vector bundles on algebraic curves

I. Moduli stacks of vector bundles

Moduli Problem [Mathematics]

A Case Study: Moduli of vector bundles on algebraic curves

- **Moduli Functor**
  $X/\mathbb{F}_q =$ smooth projective curve of genus g over the field $\mathbb{F}_q$

I. Moduli stacks of vector bundles

Moduli Problem [Mathematics]

A Case Study: Moduli of vector bundles on algebraic curves

- **Moduli Functor**

  $X/\mathbb{F}_q$ = smooth projective curve of genus g over the field $\mathbb{F}_q$
  $\mathcal{M}_X^{n,d}$ = contravariant functor, i.e. **presheaf of sets**

$$\mathcal{M}_X^{n,d} : (Sch/\mathbb{F}_q)^{op} \to (Sets)$$

I. Moduli stacks of vector bundles

Moduli Problem [Mathematics]

A Case Study: Moduli of vector bundles on algebraic curves

- **Moduli Functor**

  $X/\mathbb{F}_q$ = smooth projective curve of genus g over the field $\mathbb{F}_q$
  $\mathcal{M}_X^{n,d}$ = contravariant functor, i.e. **presheaf of sets**

$$\mathcal{M}_X^{n,d} : (Sch/\mathbb{F}_q)^{op} \to (Sets)$$

  - **(objects)** $S \mapsto \mathcal{M}_X^{n,d}(S)$ = set of iso classes of vector bundles $\mathcal{E} \downarrow X \times S$ of rank n and degree d on $X \times S$

I. Moduli stacks of vector bundles

Moduli Problem [Mathematics]

A Case Study: Moduli of vector bundles on algebraic curves

- **Moduli Functor**

  $X/\mathbb{F}_q$ = smooth projective curve of genus g over the field $\mathbb{F}_q$

  $\mathcal{M}_X^{n,d}$ = contravariant functor, i.e. **presheaf of sets**

$$\mathcal{M}_X^{n,d} : (Sch/\mathbb{F}_q)^{op} \to (Sets)$$

  - **(objects)** $S \mapsto \mathcal{M}_X^{n,d}(S)$ = set of iso classes of vector bundles $\mathcal{E} \downarrow X \times S$ of rank n and degree d on $X \times S$
  - **(morphisms)** maps of sets induced by pullback of vector bundles, i. e. $(f : S' \to S) \mapsto (f^* : \mathcal{M}_X^{n,d}(S) \to \mathcal{M}_X^{n,d}(S'))$

$$\begin{array}{ccc} (id_X \times f)^*\mathcal{E} & \longrightarrow & \mathcal{E} \\ \downarrow & & \downarrow \\ X \times S' & \xrightarrow{id_X \times f} & X \times S \end{array}$$

I. Moduli stacks of vector bundles

- **Moduli problem**
  Question: Is the moduli functor $\mathcal{M}_X^{n,d}$ **representable**, i. e. is there a scheme $M_X^{n,d}$ (= **fine moduli scheme**) s. th. for all schemes S there is a bijective correspondence of sets

$$\mathcal{M}_X^{n,d}(S) \cong Hom_{(Sch/\mathbb{F}_q)}(S, M_X^{n,d})?$$

I. Moduli stacks of vector bundles

- **Moduli problem**
  Question: Is the moduli functor $\mathcal{M}_X^{n,d}$ **representable**, i. e. is there a scheme $M_X^{n,d}$ (= **fine moduli scheme**) s. th. for all schemes S there is a bijective correspondence of sets

$$\mathcal{M}_X^{n,d}(S) \cong Hom_{(Sch/\mathbb{F}_q)}(S, M_X^{n,d})?$$

- If $M_X^{n,d}$ exists, have especially

$$\mathcal{M}_X^{n,d}(Spec(\mathbb{F}_q)) \cong Hom_{(Sch/\mathbb{F}_q)}(Spec(\mathbb{F}_q), M_X^{n,d}),$$

  i. e. iso classes of vector bundles over X correspond to points of $M_X^{n,d}$.

I. Moduli stacks of vector bundles

- If $M_X^{n,d}$ exists, have especially also

$$\mathcal{M}_X^{n,d}(M_X^{n,d}) \cong Hom_{(Sch/\mathbb{F}_q)}(M_X^{n,d}, M_X^{n,d}).$$

Let $\mathcal{E}^{univ} \downarrow X \times M_X^{n,d} \in \mathcal{M}_X^{n,d}(M_X^{n,d})$ object corresponding to morphism $\mathrm{id}_{M_X^{n,d}}$.
$\mathcal{E}^{univ} \downarrow X \times M_X^{n,d}$ **universal family** of vector bundles, i. e. for any vector bundle $\mathcal{E} \downarrow X \times S$ there is a **unique** morphism $f : S \to M_X^{n,d}$ s. th. $\mathcal{E} \cong (id_X \times f)^*(\mathcal{E}^{univ})$

$$\begin{array}{ccc} \mathcal{E} \cong (id_X \times f)^* \mathcal{E}^{univ} & \longrightarrow & \mathcal{E}^{univ} \\ \downarrow & & \downarrow \\ X \times S & \xrightarrow{id_X \times f} & X \times M_X^{n,d} \end{array}$$

I. Moduli stacks of vector bundles

- **Problem:** $\mathcal{M}_X^{n,d}$ is **not** representable, because vector bundles have non-trivial automorphisms, e. g. scalar multiplication i.e. $\mathbb{G}_m \subset Aut(\mathcal{E})$.

I. Moduli stacks of vector bundles

- **Problem:** $\mathcal{M}_X^{n,d}$ is **not** representable, because vector bundles have non-trivial automorphisms, e. g. scalar multiplication i.e. $\mathbb{G}_m \subset Aut(\mathcal{E})$.
- **Question:** Are there ways out of this dilemma?

I. Moduli stacks of vector bundles

- **Problem:** $\mathcal{M}_X^{n,d}$ is **not** representable, because vector bundles have non-trivial automorphisms, e. g. scalar multiplication i.e. $\mathbb{G}_m \subset Aut(\mathcal{E})$.
- **Question:** Are there ways out of this dilemma?
  - **Restrict** class of vector bundles to eliminate automorphisms, i. e. **rigidify** moduli problem (e. g. moduli problem for semi-stable and stable vector bundles...) and use weaker notion of representability (e. g. coarse moduli scheme...)
    Geometric Invariant Theory (GIT): Mumford, Narasimhan, Seshadri, Ramanan, Gieseker...

I. Moduli stacks of vector bundles

- **Problem:** $\mathcal{M}_X^{n,d}$ is **not** representable, because vector bundles have non-trivial automorphisms, e. g. scalar multiplication i.e. $\mathbb{G}_m \subset Aut(\mathcal{E})$.
- **Question:** Are there ways out of this dilemma?
  - **Restrict** class of vector bundles to eliminate automorphisms, i. e. **rigidify** moduli problem (e. g. moduli problem for semi-stable and stable vector bundles...) and use weaker notion of representability (e. g. coarse moduli scheme...)
    Geometric Invariant Theory (GIT): Mumford, Narasimhan, Seshadri, Ramanan, Gieseker...
  - Record information about automorphisms by organizing moduli data differently, i. e. **enlarge** category of schemes to ensure representability
    Algebraic stacks: Grothendieck, Giraud, Deligne-Mumford, Artin,...

I. Moduli stacks of vector bundles

- **Problem:** $\mathcal{M}_X^{n,d}$ is **not** representable, because vector bundles have non-trivial automorphisms, e. g. scalar multiplication i.e. $\mathbb{G}_m \subset Aut(\mathcal{E})$.
- **Question:** Are there ways out of this dilemma?
  - **Restrict** class of vector bundles to eliminate automorphisms, i. e. **rigidify** moduli problem (e. g. moduli problem for semi-stable and stable vector bundles...) and use weaker notion of representability (e. g. coarse moduli scheme...)
    Geometric Invariant Theory (GIT): Mumford, Narasimhan, Seshadri, Ramanan, Gieseker...
  - Record information about automorphisms by organizing moduli data differently, i. e. **enlarge** category of schemes to ensure representability
    Algebraic stacks: Grothendieck, Giraud, Deligne-Mumford, Artin,...

Slogan. Using stacks gives a categorification of the moduli problem!

I. Moduli stacks of vector bundles

- **Moduli stacks**
  $X/\mathbb{F}_q$ = smooth projective curve of genus g over the field $\mathbb{F}_q$

I. Moduli stacks of vector bundles

- **Moduli stacks**
  $X/\mathbb{F}_q =$ smooth projective curve of genus g over the field $\mathbb{F}_q$
  $Bun_X^{n,d} =$ **presheaf of groupoids**, i. e. **lax 2-functor, pseudofunctor**

$$Bun_X^{n,d} : (Sch/\mathbb{F}_q)^{op} \to (Groupoids)$$

I. Moduli stacks of vector bundles

- **Moduli stacks**
  $X/\mathbb{F}_q$ = smooth projective curve of genus g over the field $\mathbb{F}_q$
  $Bun_X^{n,d}$ = **presheaf of groupoids**, i. e. **lax 2-functor, pseudofunctor**

$$Bun_X^{n,d} : (Sch/\mathbb{F}_q)^{op} \to (Groupoids)$$

  - **(objects)** $S \mapsto Bun_X^{n,d}(S)$ = category with

I. Moduli stacks of vector bundles

- **Moduli stacks**
  $X/\mathbb{F}_q$ = smooth projective curve of genus g over the field $\mathbb{F}_q$
  $Bun_X^{n,d}$ = **presheaf of groupoids**, i. e. **lax 2-functor, pseudofunctor**

$$Bun_X^{n,d} : (Sch/\mathbb{F}_q)^{op} \to (Groupoids)$$

  - **(objects)** $S \mapsto Bun_X^{n,d}(S)$ = category with
    - **objects:** vector bundles (= locally free sheaves) $\mathcal{E} \downarrow X \times S$ of rank n and degree d on $X \times S$

I. Moduli stacks of vector bundles

- **Moduli stacks**
  $X/\mathbb{F}_q$ = smooth projective curve of genus g over the field $\mathbb{F}_q$
  $Bun_X^{n,d}$ = **presheaf of groupoids**, i. e. **lax 2-functor, pseudofunctor**

$$Bun_X^{n,d} : (Sch/\mathbb{F}_q)^{op} \to (Groupoids)$$

  - **(objects)** $S \mapsto Bun_X^{n,d}(S)$ = category with
    - **objects:** vector bundles (= locally free sheaves) $\mathcal{E} \downarrow X \times S$ of rank n and degree d on $X \times S$
    - **morphisms:** vector bundle isomorphisms

I. Moduli stacks of vector bundles

- **Moduli stacks**
  $X/\mathbb{F}_q$ = smooth projective curve of genus g over the field $\mathbb{F}_q$
  $Bun_X^{n,d}$ = **presheaf of groupoids**, i. e. **lax 2-functor, pseudofunctor**

$$Bun_X^{n,d} : (Sch/\mathbb{F}_q)^{op} \to (Groupoids)$$

  - **(objects)** $S \mapsto Bun_X^{n,d}(S)$ = category with
    - **objects:** vector bundles (= locally free sheaves) $\mathcal{E} \downarrow X \times S$ of rank n and degree d on $X \times S$
    - **morphisms:** vector bundle isomorphisms
  - **(1-morphisms)** functors induced by pullbacks of vector bundles, i. e. $(f : S' \to S) \mapsto (f^* : Bun_X^{n,d}(S) \to Bun_X^{n,d}(S'))$

I. Moduli stacks of vector bundles

- **Moduli stacks**
  $X/\mathbb{F}_q$ = smooth projective curve of genus g over the field $\mathbb{F}_q$
  $Bun_X^{n,d}$ = **presheaf of groupoids**, i. e. **lax 2-functor, pseudofunctor**

$$Bun_X^{n,d} : (Sch/\mathbb{F}_q)^{op} \to (Groupoids)$$

  - **(objects)** $S \mapsto Bun_X^{n,d}(S)$ = category with
    - **objects:** vector bundles (= locally free sheaves) $\mathcal{E} \downarrow X \times S$ of rank n and degree d on $X \times S$
    - **morphisms:** vector bundle isomorphisms
  - **(1-morphisms)** functors induced by pullbacks of vector bundles, i. e. $(f : S' \to S) \mapsto (f^* : Bun_X^{n,d}(S) \to Bun_X^{n,d}(S'))$
  - **(2-morphisms)** natural isomorphisms between pullback functors, i. e. $(S'' \xrightarrow{g} S' \xrightarrow{f} S) \mapsto (\epsilon_{f,g} : g^* \circ f^* \cong (f \circ g)^*)$

I. Moduli stacks of vector bundles

Question: We like to glue local data so need a ’sheaf of groupoids’.

I. Moduli stacks of vector bundles

Question: We like to glue local data so need a ’sheaf of groupoids’.

Grothendieck topology

I. Moduli stacks of vector bundles

Question: We like to glue local data so need a 'sheaf of groupoids'.

Grothendieck topology

- a category $\mathcal{C}$: objects U "open sets"

I. Moduli stacks of vector bundles

Question: We like to glue local data so need a 'sheaf of groupoids'.

Grothendieck topology

- a category $\mathcal{C}$: objects U "open sets"
- a collection of morphisms in $\mathcal{C}$ with same target: $\{U_i \to U\}_{i \in I}$ "coverings" s. th.

I. Moduli stacks of vector bundles

Question: We like to glue local data so need a 'sheaf of groupoids'.

Grothendieck topology

- a category $\mathcal{C}$: objects U "open sets"
- a collection of morphisms in $\mathcal{C}$ with same target: $\{U_i \to U\}_{i \in I}$ "coverings" s. th.
    - **Isomorphisms** Every isomorphism $\{U' \stackrel{\cong}{\to} U\}$ is a covering.

I. Moduli stacks of vector bundles

Question: We like to glue local data so need a 'sheaf of groupoids'.

Grothendieck topology

- a category $\mathcal{C}$: objects U "open sets"
- a collection of morphisms in $\mathcal{C}$ with same target: $\{U_i \to U\}_{i \in I}$ "coverings" s. th.
  - **Isomorphisms** Every isomorphism $\{U' \overset{\cong}{\to} U\}$ is a covering.
  - **Transitivity** Coverings of coverings are coverings, i. e. if $\{U_i \overset{p_i}{\to} U\}$ covering and $\{U_{ij} \overset{p_{ij}}{\to} U_i\}$ covering, then also $\{U_{ij} \overset{p_i \circ p_{ij}}{\to} U\}$ covering.

I. Moduli stacks of vector bundles

Question: We like to glue local data so need a 'sheaf of groupoids'.

Grothendieck topology

- a category $\mathcal{C}$: objects U "open sets"
- a collection of morphisms in $\mathcal{C}$ with same target: $\{U_i \to U\}_{i \in I}$ "coverings" s. th.
  - **Isomorphisms** Every isomorphism $\{U' \stackrel{\cong}{\to} U\}$ is a covering.
  - **Transitivity** Coverings of coverings are coverings, i. e. if $\{U_i \stackrel{p_i}{\to} U\}$ covering and $\{U_{ij} \stackrel{p_{ij}}{\to} U_i\}$ covering, then also $\{U_{ij} \stackrel{p_i \circ p_{ij}}{\to} U\}$ covering.
  - **Base change** Coverings respect base change, i. e. if $\{U_i \stackrel{p_i}{\to} U\}$ covering, $V \to U$ morphism, then $\{V \times_U U_i \to V\}$ covering.

I. Moduli stacks of vector bundles

Question: We like to glue local data so need a 'sheaf of groupoids'.

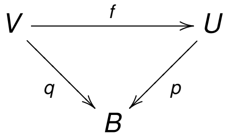
Grothendieck topology

- a category $\mathcal{C}$: objects U "open sets"
- a collection of morphisms in $\mathcal{C}$ with same target: $\{U_i \to U\}_{i\in I}$ "coverings" s. th.
  - **Isomorphisms** Every isomorphism $\{U' \stackrel{\cong}{\to} U\}$ is a covering.
  - **Transitivity** Coverings of coverings are coverings, i. e. if $\{U_i \stackrel{p_i}{\to} U\}$ covering and $\{U_{ij} \stackrel{p_{ij}}{\to} U_i\}$ covering, then also $\{U_{ij} \stackrel{p_i \circ p_{ij}}{\to} U\}$ covering.
  - **Base change** Coverings respect base change, i. e. if $\{U_i \stackrel{p_i}{\to} U\}$ covering, $V \to U$ morphism, then $\{V \times_U U_i \to V\}$ covering.

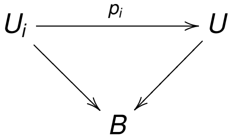
A category $\mathcal{C}$ with a Grothendieck topology is a **site** denoted by $\mathcal{C}_\tau$.

I. Moduli stacks of vector bundles

- **Example. Smooth topology on category of schemes**
  $(Sch/B)_{sm}$ = **smooth site** of schemes over a fixed base scheme B
  - **objects:** $p : U \to B$ smooth morphisms of schemes
  - **morphisms:** commutative diagrams of the form



  - **coverings:** commutative diagrams of the form



    s. th. $U = \coprod_{i \in I} p_i(U_i)$

I. Moduli stacks of vector bundles

Definition

A **stack** is a sheaf of groupoids over the smooth site $(Sch/B)_{sm}$, i. e. a presheaf

$$\mathfrak{X} : (Sch/B)^{op} \to (Groupoids)$$

satisfying sheaf axioms for coverings $\{U_i \to U\}$ in $(Sch/B)_{sm}$:

I. Moduli stacks of vector bundles

Definition

A **stack** is a sheaf of groupoids over the smooth site $(Sch/B)_{sm}$, i. e. a presheaf

$$\mathfrak{X} : (Sch/B)^{op} \to (Groupoids)$$

satisfying sheaf axioms for coverings $\{U_i \to U\}$ in $(Sch/B)_{sm}$:

- **(Glueing of objects)**
  If X_i are objects of $\mathfrak{X}(U_i)$, $\phi_{ij} : X_j|U_i \times_U U_j \to X_i|U_i \times_U U_j$ morphisms satisfying the cocycle condition

  $$\phi_{ij}|U_i \times_U U_j \times_U U_k \circ \phi_{jk}|U_i \times_U U_j \times_U U_k = \phi_{ik}|U_i \times_U U_j \times_U U_k$$

  then there exists object X of $\mathfrak{X}(U)$ and morphisms $\phi_i : X|U_i \xrightarrow{\cong} X_i$ with

  $$\phi_{ji} \circ \phi_i|U_i \times_U U_j = \phi_j|U_i \times_U U_j$$

I. Moduli stacks of vector bundles

Definition (contin.)

- **(Glueing of morphisms)**
  If X and X' are objects of $\mathfrak{X}(U)$, $\phi_i : X|U_i \to X'|U_i$ morphisms with

$$\phi_i|U_i \times_U U_j = \phi_j|U_i \times_U U_j$$

  then there exists morphism $\eta : X \to X'$ with

$$\eta|U_i = \phi_i.$$

I. Moduli stacks of vector bundles

Definition (contin.)

- **(Glueing of morphisms)**
  If X and X' are objects of $\mathfrak{X}(U)$, $\phi_i : X|U_i \to X'|U_i$ morphisms with
  $$\phi_i|U_i \times_U U_j = \phi_j|U_i \times_U U_j$$
  then there exists morphism $\eta : X \to X'$ with
  $$\eta|U_i = \phi_i.$$
- **(Monopresheaf)**
  If X and X' are objects of $\mathfrak{X}(U)$, $\phi : X \to X'$, $\psi : X \to X'$ morphisms with
  $$\phi|U_i = \psi|U_i$$
  then $\phi = \psi$.

I. Moduli stacks of vector bundles

- **(Morphisms of stacks)**
  $F : \mathfrak{X} \to \mathfrak{Y}$ given by functors $F_S^* : \mathfrak{Y}(S) \to \mathfrak{X}(S)$ for any S and natural transformations $F_f^* : f^* \circ F_S^* \overset{\cong}{\to} F_{S'}^* \circ f^*$ for any $f : S' \to S$.

I. Moduli stacks of vector bundles

- **(Morphisms of stacks)**
  $F : \mathfrak{X} \to \mathfrak{Y}$ given by functors $F^*_S : \mathfrak{Y}(S) \to \mathfrak{X}(S)$ for any S and natural transformations $F^*_f : f^* \circ F^*_S \overset{\cong}{\to} F^*_{S'} \circ f^*$ for any $f : S' \to S$.
- (**Every sheaf is a stack**)
  Any sheaf $F : (Sch/B)^{op} \to (Sets)$ is a stack by considering the sets $F(S)$ as groupoids.

I. Moduli stacks of vector bundles

- **(Morphisms of stacks)**
  $F : \mathfrak{X} \to \mathfrak{Y}$ given by functors $F_S^* : \mathfrak{Y}(S) \to \mathfrak{X}(S)$ for any S and natural transformations $F_f^* : f^* \circ F_S^* \stackrel{\cong}{\to} F_{S'}^* \circ f^*$ for any $f : S' \to S$.
- (**Every sheaf is a stack**)
  Any sheaf $F : (Sch/B)^{op} \to (Sets)$ is a stack by considering the sets $F(S)$ as groupoids.
  (Grothendieck) Any scheme S is a stack given as the sheaf $\underline{S} := Hom_{(Sch/B)}(-, S)$.

I. Moduli stacks of vector bundles

- **(Morphisms of stacks)**
  $F : \mathfrak{X} \to \mathfrak{Y}$ given by functors $F_S^* : \mathfrak{Y}(S) \to \mathfrak{X}(S)$ for any S and natural transformations $F_f^* : f^* \circ F_S^* \stackrel{\cong}{\to} F_{S'}^* \circ f^*$ for any $f : S' \to S$.
- (**Every sheaf is a stack**)
  Any sheaf $F : (Sch/B)^{op} \to (Sets)$ is a stack by considering the sets $F(S)$ as groupoids.
  (Grothendieck) Any scheme S is a stack given as the sheaf $\underline{S} := Hom_{(Sch/B)}(-, S)$.
- **(Representable morphisms of stacks)**
  $F : \mathfrak{X} \to \mathfrak{Y}$ **representable** if for any morphism $Y \to \mathfrak{Y}$ the fibred product $\mathfrak{X} \times_{\mathfrak{Y}} Y$ is a stack isomorphic to a scheme S, i. e. to the stack $\underline{S} = Hom_{(Sch/B)}(-, S)$

$$\begin{array}{ccc} \mathfrak{X} \times_{\mathfrak{Y}} Y & \longrightarrow & \mathfrak{X} \\ \downarrow & & \downarrow \\ Y & \longrightarrow & \mathfrak{Y} \end{array}$$

I. Moduli stacks of vector bundles

- **(2-Yoneda Lemma)[Giraud, Hakim]**
  Let $\mathfrak{X}$ be a stack over (Sch/B) and S a scheme. There is an equivalence of categories

$$\theta : Hom_{Stacks}(S, \mathfrak{X}) \xrightarrow{\cong} \mathfrak{X}(S), \ (f : S \to \mathfrak{X}) \mapsto f(id_S)$$

I. Moduli stacks of vector bundles

- **(2-Yoneda Lemma)[Giraud, Hakim]**
  Let $\mathfrak{X}$ be a stack over (Sch/B) and S a scheme. There is an equivalence of categories

$$\theta : Hom_{Stacks}(S, \mathfrak{X}) \overset{\cong}{\to} \mathfrak{X}(S), \ (f : S \to \mathfrak{X}) \mapsto f(id_S)$$

Definition

A stack over the smooth site $(Sch/B)_{sm}$

$$\mathfrak{X} : (Sch/B)^{op} \to (Groupoids)$$

is an **algebraic stack** if

- $\Delta : \mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ is representable, quasi-compact and separated

I. Moduli stacks of vector bundles

- **(2-Yoneda Lemma)[Giraud, Hakim]**
  Let $\mathfrak{X}$ be a stack over (Sch/B) and S a scheme. There is an equivalence of categories

$$\theta : Hom_{Stacks}(S, \mathfrak{X}) \stackrel{\cong}{\to} \mathfrak{X}(S),\ (f : S \to \mathfrak{X}) \mapsto f(id_S)$$

Definition

A stack over the smooth site $(Sch/B)_{sm}$

$$\mathfrak{X} : (Sch/B)^{op} \to (Groupoids)$$

is an **algebraic stack** if

- $\Delta : \mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ is representable, quasi-compact and separated
- there is a scheme X (**atlas**) together with smooth surjective morphism $x : X \to \mathfrak{X}$

I. Moduli stacks of vector bundles

Quotient Problems.

I. Moduli stacks of vector bundles

Quotient Problems.

- X = smooth (noetherian) scheme over $\mathbb{F}_q$, G = smooth affine algebraic group over $\mathbb{F}_q$ with a **free** action $\rho : G \times X \to X$.

I. Moduli stacks of vector bundles

Quotient Problems.

- X = smooth (noetherian) scheme over $\mathbb{F}_q$, G = smooth affine algebraic group over $\mathbb{F}_q$ with a **free** action $\rho : G \times X \to X$. Then we have:
  - (i) The quotient X/G exists as a scheme and the quotient morphism $\tau : X \to X/G$ is a principal G-bundle.

I. Moduli stacks of vector bundles

Quotient Problems.

- X = smooth (noetherian) scheme over $\mathbb{F}_q$, G = smooth affine algebraic group over $\mathbb{F}_q$ with a **free** action $\rho : G \times X \to X$.
  Then we have:
  (i) The quotient X/G exists as a scheme and the quotient morphism $\tau : X \to X/G$ is a principal G-bundle.
  (ii) The points of X/G are given by morphisms of schemes $S \to X/G$ and for any such morphism $s : S \to X/G$ have pullback

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{\mu} & X \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \tau} \\ S & \xrightarrow{s} & X/G \end{array}$$

I. Moduli stacks of vector bundles

Quotient Problems.

- $X =$ smooth (noetherian) scheme over $\mathbb{F}_q$, $G =$ smooth affine algebraic group over $\mathbb{F}_q$ with a **free** action $\rho : G \times X \to X$.
  Then we have:
  (i) The quotient X/G exists as a scheme and the quotient morphism $\tau : X \to X/G$ is a principal G-bundle.
  (ii) The points of X/G are given by morphisms of schemes $S \to X/G$ and for any such morphism $s : S \to X/G$ have pullback

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{\mu} & X \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \tau} \\ S & \xrightarrow{s} & X/G \end{array}$$

  So, s defines a principal G-bundle $\pi : \mathcal{P} \to S$ plus G-equivariant morphism $\mu : \mathcal{P} \to X$.

I. Moduli stacks of vector bundles

Quotient Problems.

- X = smooth (noetherian) scheme over $\mathbb{F}_q$, G = smooth affine algebraic group over $\mathbb{F}_q$ with a **free** action $\rho : G \times X \to X$.
  Then we have:
  (i) The quotient X/G exists as a scheme and the quotient morphism $\tau : X \to X/G$ is a principal G-bundle.
  (ii) The points of X/G are given by morphisms of schemes $S \to X/G$ and for any such morphism $s : S \to X/G$ have pullback

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{\mu} & X \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \tau} \\ S & \xrightarrow{s} & X/G \end{array}$$

  So, s defines a principal G-bundle $\pi : \mathcal{P} \to S$ plus G-equivariant morphism $\mu : \mathcal{P} \to X$.
- **Question:** What happens if the action is **not** free?

I. Moduli stacks of vector bundles

- $X =$ smooth (noetherian) scheme over $\mathbb{F}_q$, $G =$ smooth affine algebraic group over $\mathbb{F}_q$ with an action $\rho : G \times X \to X$
  $[X/G] =$ **quotient stack** over smooth site $(Sch/\mathbb{F}_q)_{sm}$

$$[X/G] : (Sch/\mathbb{F}_q)^{op} \to (Groupoids)$$

I. Moduli stacks of vector bundles

- X = smooth (noetherian) scheme over $\mathbb{F}_q$, G = smooth affine algebraic group over $\mathbb{F}_q$ with an action $\rho : G \times X \to X$
  $[X/G]$ = **quotient stack** over smooth site $(Sch/\mathbb{F}_q)_{sm}$

$$[X/G] : (Sch/\mathbb{F}_q)^{op} \to (Groupoids)$$

  - **(objects)** $S \mapsto [X/G](S)$ = category with
    - **objects:** principal G-bundles $\pi : \mathcal{P} \downarrow S$ together with G-equivariant morphism $\mu : \mathcal{P} \to X$
    - **morphisms:** isomorphisms of principal G-bundles commuting with G-equivariant morphisms

I. Moduli stacks of vector bundles

- X = smooth (noetherian) scheme over $\mathbb{F}_q$, G = smooth affine algebraic group over $\mathbb{F}_q$ with an action $\rho : G \times X \to X$
  $[X/G]$ = **quotient stack** over smooth site $(Sch/\mathbb{F}_q)_{sm}$

$$[X/G] : (Sch/\mathbb{F}_q)^{op} \to (Groupoids)$$

  - **(objects)** $S \mapsto [X/G](S)$ = category with
    - **objects:** principal G-bundles $\pi : \mathcal{P} \downarrow S$ together with G-equivariant morphism $\mu : \mathcal{P} \to X$
    - **morphisms:** isomorphisms of principal G-bundles commuting with G-equivariant morphisms
  - **(1-morphism)** functors induced by pullbacks of principal G-bundles bundles, i. e.
    $(f : S' \to S) \mapsto (f^* : [X/G](S) \to [X/G](S'))$

I. Moduli stacks of vector bundles

- $X =$ smooth (noetherian) scheme over $\mathbb{F}_q$, $G =$ smooth affine algebraic group over $\mathbb{F}_q$ with an action $\rho : G \times X \to X$
  $[X/G] =$ **quotient stack** over smooth site $(Sch/\mathbb{F}_q)_{sm}$

$$[X/G] : (Sch/\mathbb{F}_q)^{op} \to (Groupoids)$$

  - **(objects)** $S \mapsto [X/G](S) =$ category with
    - **objects:** principal G-bundles $\pi : \mathcal{P} \downarrow S$ together with G-equivariant morphism $\mu : \mathcal{P} \to X$
    - **morphisms:** isomorphisms of principal G-bundles commuting with G-equivariant morphisms
  - **(1-morphism)** functors induced by pullbacks of principal G-bundles bundles, i. e.
    $(f : S' \to S) \mapsto (f^* : [X/G](S) \to [X/G](S'))$
  - **(2-morphisms)** natural isomorphism between pullback functors i. e.
    $(S'' \xrightarrow{g} S' \xrightarrow{f} S) \mapsto (\epsilon_{f,g} : g^* \circ f^* \cong (f \circ g)^*)$

I. Moduli stacks of vector bundles

- $X =$ smooth (noetherian) scheme over $\mathbb{F}_q$, $G =$ smooth affine algebraic group over $\mathbb{F}_q$ with an action $\rho : G \times X \to X$
  $[X/G] =$ **quotient stack** over smooth site $(Sch/\mathbb{F}_q)_{sm}$

$$[X/G] : (Sch/\mathbb{F}_q)^{op} \to (Groupoids)$$

  - **(objects)** $S \mapsto [X/G](S) =$ category with
    - **objects:** principal G-bundles $\pi : \mathcal{P} \downarrow S$ together with G-equivariant morphism $\mu : \mathcal{P} \to X$
    - **morphisms:** isomorphisms of principal G-bundles commuting with G-equivariant morphisms
  - **(1-morphism)** functors induced by pullbacks of principal G-bundles bundles, i. e.
    $(f : S' \to S) \mapsto (f^* : [X/G](S) \to [X/G](S'))$
  - **(2-morphisms)** natural isomorphism between pullback functors i. e.
    $(S'' \xrightarrow{g} S' \xrightarrow{f} S) \mapsto (\epsilon_{f,g} : g^* \circ f^* \cong (f \circ g)^*)$
- if $X = Spec(\mathbb{F}_q)$, i.e. point with trivial G-action, then $[Spec(\mathbb{F}_q)/G]$ is **classifying stack** $\mathcal{B}G$ of all principal G-bundles

I. Moduli stacks of vector bundles

Theorem

$[X/G]$ *is an algebraic stack.*

I. Moduli stacks of vector bundles

Theorem

$[X/G]$ is an algebraic stack.

Proof.

- **Construction of atlas x.** Trivial G-bundle $G \times X \downarrow X$ with action $\rho : G \times X \to X$ gives object in groupoid $[X/G](X)$, i. e. defines a morphism of stacks $x : X \to [X/G]$.

I. Moduli stacks of vector bundles

Theorem

$[X/G]$ is an algebraic stack.

Proof.

- **Construction of atlas x.** Trivial G-bundle $G \times X \downarrow X$ with action $\rho : G \times X \to X$ gives object in groupoid $[X/G](X)$, i. e. defines a morphism of stacks $x : X \to [X/G]$.
- **Properties of x.** For any S and any $s : S \to [X/G]$ let $\pi : \mathcal{P} \downarrow S$ be the corresponding principal G-bundle with G-equivariant morphism $\mu : \mathcal{P} \to X$, then $S \times_{[X/G]} X \cong \mathcal{P}$. x surjective, smooth, because π surjective, smooth for every s.

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{\mu} & X \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle x} \\ S & \xrightarrow{s} & [X/G] \end{array}$$

□

I. Moduli stacks of vector bundles

- $X/\mathbb{F}_q =$ smooth projective curve of genus g over the field $\mathbb{F}_q$

I. Moduli stacks of vector bundles

- $X/\mathbb{F}_q =$ smooth projective curve of genus g over the field $\mathbb{F}_q$
- $Bun_X^{n,d} =$ **moduli stack** of vector bundles of rank n and degree d on X over the smooth site $(Sch/\mathbb{F}_q)_{sm}$ of schemes over $Spec(\mathbb{F}_q)$

$$Bun_X^{n,d} : (Sch/\mathbb{F}_q)^{op} \to (Groupoids)$$

I. Moduli stacks of vector bundles

- $X/\mathbb{F}_q$ = smooth projective curve of genus g over the field $\mathbb{F}_q$
- $Bun_X^{n,d}$ = **moduli stack** of vector bundles of rank n and degree d on X over the smooth site $(Sch/\mathbb{F}_q)_{sm}$ of schemes over $Spec(\mathbb{F}_q)$

$$Bun_X^{n,d} : (Sch/\mathbb{F}_q)^{op} \to (Groupoids)$$

  - **(objects)** $S \mapsto Bun_X^{n,d}(S)$ = category with
    - **objects:** vector bundles (= locally free sheaves) $\mathcal{E} \downarrow X \times S$ of rank n and degree d on $X \times S$
    - **morphisms:** vector bundle isomorphisms

I. Moduli stacks of vector bundles

- $X/\mathbb{F}_q$ = smooth projective curve of genus g over the field $\mathbb{F}_q$
- $Bun_X^{n,d}$ = **moduli stack** of vector bundles of rank n and degree d on X over the smooth site $(Sch/\mathbb{F}_q)_{sm}$ of schemes over $Spec(\mathbb{F}_q)$

$$Bun_X^{n,d} : (Sch/\mathbb{F}_q)^{op} \to (Groupoids)$$

  - **(objects)** $S \mapsto Bun_X^{n,d}(S)$ = category with
    - **objects:** vector bundles (= locally free sheaves) $\mathcal{E} \downarrow X \times S$ of rank n and degree d on $X \times S$
    - **morphisms:** vector bundle isomorphisms
  - **(1-morphisms)** functors induced by pullbacks of vector bundles $(f : S' \to S) \mapsto (f^* : Bun_X^{n,d}(S) \to Bun_X^{n,d}(S'))$

I. Moduli stacks of vector bundles

- $X/\mathbb{F}_q$ = smooth projective curve of genus g over the field $\mathbb{F}_q$
- $Bun_X^{n,d}$ = **moduli stack** of vector bundles of rank n and degree d on X over the smooth site $(Sch/\mathbb{F}_q)_{sm}$ of schemes over $Spec(\mathbb{F}_q)$

$$Bun_X^{n,d} : (Sch/\mathbb{F}_q)^{op} \to (Groupoids)$$

- **(objects)** $S \mapsto Bun_X^{n,d}(S)$ = category with
  - **objects:** vector bundles (= locally free sheaves) $\mathcal{E} \downarrow X \times S$ of rank n and degree d on $X \times S$
  - **morphisms:** vector bundle isomorphisms
- **(1-morphisms)** functors induced by pullbacks of vector bundles $(f : S' \to S) \mapsto (f^* : Bun_X^{n,d}(S) \to Bun_X^{n,d}(S'))$
- **(2-morphisms)** natural isomorphisms between pullback functors $(S'' \stackrel{g}{\to} S' \stackrel{f}{\to} S) \mapsto (\epsilon_{f,g} : g^* \circ f^* \cong (f \circ g)^*$

I. Moduli stacks of vector bundles

- $X/\mathbb{F}_q$ = smooth projective curve of genus g over the field $\mathbb{F}_q$
- $Bun_X^{n,d}$ = **moduli stack** of vector bundles of rank n and degree d on X over the smooth site $(Sch/\mathbb{F}_q)_{sm}$ of schemes over $Spec(\mathbb{F}_q)$

$$Bun_X^{n,d} : (Sch/\mathbb{F}_q)^{op} \to (Groupoids)$$

- **(objects)** $S \mapsto Bun_X^{n,d}(S)$ = category with
  - **objects:** vector bundles (= locally free sheaves) $\mathcal{E} \downarrow X \times S$ of rank n and degree d on $X \times S$
  - **morphisms:** vector bundle isomorphisms
- **(1-morphisms)** functors induced by pullbacks of vector bundles $(f : S' \to S) \mapsto (f^* : Bun_X^{n,d}(S) \to Bun_X^{n,d}(S'))$
- **(2-morphisms)** natural isomorphisms between pullback functors $(S'' \xrightarrow{g} S' \xrightarrow{f} S) \mapsto (\epsilon_{f,g} : g^* \circ f^* \cong (f \circ g)^*$

Theorem

$Bun_X^{n,d}$ is an algebraic stack, smooth and locally of finite type.

I. Moduli stacks of vector bundles

Question: Why using the language of algebraic stacks?

I. Moduli stacks of vector bundles

Question: Why using the language of algebraic stacks?

- Moduli problem for vector bundles has no fine solution in schemes, but in stacks, i.e. the functor $Bun_X^{n,d}$ is **representable** in stacks. i.e. there is an equivalence of categories for any scheme S over $\mathbb{F}_q$
$$Bun_X^{n,d}(S) \cong Hom_{Stacks}(\underline{S}, Bun_X^{n,d})$$
where $\underline{S} := Hom_{Sch/\mathbb{F}_q}(-, S)$ is the **stack associated to** S. **(2-Yoneda Lemma)**

I. Moduli stacks of vector bundles

Question: Why using the language of algebraic stacks?

- Moduli problem for vector bundles has no fine solution in schemes, but in stacks, i.e. the functor $Bun_X^{n,d}$ is **representable** in stacks. i.e. there is an equivalence of categories for any scheme S over $\mathbb{F}_q$

$$Bun_X^{n,d}(S) \cong Hom_{Stacks}(\underline{S}, Bun_X^{n,d})$$

  where $\underline{S} := Hom_{Sch/\mathbb{F}_q}(-, S)$ is the **stack associated to** S. **(2-Yoneda Lemma)**

- There exists a **universal vector bundle** $\mathcal{E}^{univ} \downarrow X \times Bun_X^{n,d}$ s. th. for any vector bundle $\mathcal{F} \downarrow X \times S$ there is a morphism of stacks

$$\varphi : \underline{S} \to Bun_X^{n,d}$$

  s. th. $\mathcal{F}$ is given via the pullback

$$\mathcal{F} \cong (id_X \times \varphi)^*(\mathcal{E}^{univ})$$

II. Cohomology of moduli stacks of vector bundles

Question: How to define cohomology of an algebraic stack $\mathfrak{X}$?
General formalism for cohomology of stacks:
Deligne-Mumford, Behrend, Laumon-Moret-Bailly, Laszlo-Olsson ...

II. Cohomology of moduli stacks of vector bundles

Question: How to define cohomology of an algebraic stack $\mathfrak{X}$?
General formalism for cohomology of stacks:
Deligne-Mumford, Behrend, Laumon-Moret-Bailly, Laszlo-Olsson ...

$\mathfrak{X}_{sm} =$ **smooth site** of an algebraic stack $\mathfrak{X}$ i. e.

- **objects:** smooth morphism $X \to \mathfrak{X}$ with X a scheme
- **morphisms:** commutative diagrams

$$\begin{array}{ccc} X & \longrightarrow & Y \\ & \searrow \quad \swarrow & \\ & \mathfrak{X} & \end{array}$$

- **coverings:** smooth coverings of the schemes

II. Cohomology of moduli stacks of vector bundles

$\mathcal{F}$ = sheaf on $\mathfrak{X}_{sm}$ is given by

II. Cohomology of moduli stacks of vector bundles

$\mathcal{F}$ = sheaf on $\mathfrak{X}_{sm}$ is given by

- for each atlas $x : X \to \mathfrak{X}$ a sheaf $\mathcal{F}_X$ on X

II. Cohomology of moduli stacks of vector bundles

$\mathcal{F}$ = sheaf on $\mathfrak{X}_{sm}$ is given by

- for each atlas $x : X \to \mathfrak{X}$ a sheaf $\mathcal{F}_X$ on X
- for each commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle x}\searrow \quad \swarrow{\scriptstyle y} & \\ & \mathfrak{X} & \end{array}$$

  an isomorphism

$$\varphi_f : \mathcal{F}_X \xrightarrow{\cong} f^*\mathcal{F}_Y$$

  satisfying cocycle condition for 3 morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$

$$\varphi_{g \circ f} = \varphi_f \circ f^*\varphi_g : \mathcal{F}_X \xrightarrow{\cong} (g \circ f)^*\mathcal{F}_Z \cong f^*(g^*\mathcal{F}_Z)$$

II. Cohomology of moduli stacks of vector bundles

$\mathcal{F}$ = sheaf on $\mathfrak{X}_{sm}$ is given by

- for each atlas $x : X \to \mathfrak{X}$ a sheaf $\mathcal{F}_X$ on X
- for each commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ & {\scriptstyle x}\searrow \quad \swarrow{\scriptstyle y} & \\ & \mathfrak{X} & \end{array}$$

  an isomorphism

$$\varphi_f : \mathcal{F}_X \overset{\cong}{\to} f^*\mathcal{F}_Y$$

  satisfying cocycle condition for 3 morphisms $X \overset{f}{\to} Y \overset{g}{\to} Z$

$$\varphi_{g \circ f} = \varphi_f \circ f^*\varphi_g : \mathcal{F}_X \overset{\cong}{\to} (g \circ f)^*\mathcal{F}_Z \cong f^*(g^*\mathcal{F}_Z)$$

$(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ is a ringed site with structure sheaf $\mathcal{O}_{\mathfrak{X}}$ given by assembly of structure sheaves $\mathcal{O}_X$ on atlas $x : X \to \mathfrak{X}$

II. Cohomology of moduli stacks of vector bundles

Let $\mathcal{F}$ be sheaf of abelian groups on the site $\mathfrak{X}_{sm}$. The **global sections** are defined as:

$$\Gamma(\mathfrak{X}, \mathcal{F}) := \varprojlim \Gamma(X, \mathcal{F}|_X)$$

where the limit is taken over all atlases $x : X \to \mathfrak{X}$.

II. Cohomology of moduli stacks of vector bundles

Let $\mathcal{F}$ be sheaf of abelian groups on the site $\mathfrak{X}_{sm}$. The **global sections** are defined as:

$$\Gamma(\mathfrak{X}, \mathcal{F}) := \varprojlim \Gamma(X, \mathcal{F}|_X)$$

where the limit is taken over all atlases $x : X \to \mathfrak{X}$.

Definition

The **smooth cohomology** of the algebraic stack $\mathfrak{X}$ with respect to a sheaf $\mathcal{F}$ of abelian groups on the smooth site $\mathfrak{X}_{sm}$ is defined as

$$H^i_{sm}(\mathfrak{X}, \mathcal{F}) := R^i\Gamma(\mathfrak{X}, \mathcal{F})$$

II. Cohomology of moduli stacks of vector bundles

Let $X_\bullet = \{X_i\}_{i \geq 0}$ be the associated simplicial scheme with

$$X_i := X \times_{\mathfrak{X}} X \times_{\mathfrak{X}} \cdots \times_{\mathfrak{X}} X$$

get spectral sequence

$$E_1^{p,q} \cong H^p_{sm}(X_q, \mathcal{F}|_{X_q}) \Rightarrow H^{p+q}_{sm}(\mathfrak{X}, \mathcal{F})$$

II. Cohomology of moduli stacks of vector bundles

Let $X_\bullet = \{X_i\}_{i \geq 0}$ be the associated simplicial scheme with

$$X_i := X \times_{\mathfrak{X}} X \times_{\mathfrak{X}} \cdots \times_{\mathfrak{X}} X$$

get spectral sequence

$$E_1^{p,q} \cong H^p_{sm}(X_q, \mathcal{F}|_{X_q}) \Rightarrow H^{p+q}_{sm}(\mathfrak{X}, \mathcal{F})$$

Let $\mathfrak{X}$ be an algebraic stack over the field $\mathbb{F}_q$ with base change extension

$$\overline{\mathfrak{X}} = \mathfrak{X} \times_{Spec(\mathbb{F}_q)} Spec(\overline{\mathbb{F}}_q)$$

II. Cohomology of moduli stacks of vector bundles

Let $X_\bullet = \{X_i\}_{i \geq 0}$ be the associated simplicial scheme with

$$X_i := X \times_{\mathfrak{X}} X \times_{\mathfrak{X}} \cdots \times_{\mathfrak{X}} X$$

get spectral sequence

$$E_1^{p,q} \cong H^p_{sm}(X_q, \mathcal{F}|_{X_q}) \Rightarrow H^{p+q}_{sm}(\mathfrak{X}, \mathcal{F})$$

Let $\mathfrak{X}$ be an algebraic stack over the field $\mathbb{F}_q$ with base change extension

$$\overline{\mathfrak{X}} = \mathfrak{X} \times_{Spec(\mathbb{F}_q)} Spec(\overline{\mathbb{F}}_q)$$

Definition

The **l-adic cohomology** of the algebraic stack $\overline{\mathfrak{X}}$ over $\overline{\mathbb{F}}_q$ is defined as

$$H^*_{sm}(\overline{\mathfrak{X}}, \mathbb{Q}_l) := \varprojlim H^*_{sm}(\bar{\mathfrak{X}}, \mathbb{Z}/l^m\mathbb{Z}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$$

II. Cohomology of moduli stacks of vector bundles

Let $X_\bullet = \{X_i\}_{i \geq 0}$ be the associated simplicial scheme with

$$X_i := X \times_{\mathfrak{X}} X \times_{\mathfrak{X}} \cdots \times_{\mathfrak{X}} X$$

get spectral sequence

$$E_1^{p,q} \cong H_{sm}^p(X_q, \mathcal{F}|_{X_q}) \Rightarrow H_{sm}^{p+q}(\mathfrak{X}, \mathcal{F})$$

Let $\mathfrak{X}$ be an algebraic stack over the field $\mathbb{F}_q$ with base change extension

$$\overline{\mathfrak{X}} = \mathfrak{X} \times_{Spec(\mathbb{F}_q)} Spec(\overline{\mathbb{F}}_q)$$

Definition

The **l-adic cohomology** of the algebraic stack $\overline{\mathfrak{X}}$ over $\overline{\mathbb{F}}_q$ is defined as

$$H_{sm}^*(\overline{\mathfrak{X}}, \mathbb{Q}_l) := \varprojlim H_{sm}^*(\bar{\mathfrak{X}}, \mathbb{Z}/l^m\mathbb{Z}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$$

We will analyze: $H_{sm}^*(\overline{Bun}_X^{n,d}, \mathbb{Q}_l) = \varprojlim H_{sm}^*(\overline{Bun}_X^{n,d}, \mathbb{Z}/l^m\mathbb{Z}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$

II. Cohomology of moduli stacks of vector bundles

Theorem (Weil, Deligne)

Let X be a smooth projective curve of genus g over $\mathbb{F}_q$. Then:

$$H^0_{sm}(\overline{X};\mathbb{Q}_l) = \mathbb{Q}_l \cdot 1$$
$$H^1_{sm}(\overline{X};\mathbb{Q}_l) = \bigoplus_{i=1}^{2g} \mathbb{Q}_l \cdot \alpha_i$$
$$H^2_{sm}(\overline{X};\mathbb{Q}_l) = \mathbb{Q}_l \cdot [\overline{X}]$$
$$H^i_{sm}(\overline{X};\mathbb{Q}_l) = 0, \ \textit{if } i \geq 3$$

where $[\overline{X}]$ is the fundamental class and the α_i are eigenclasses under the action of the geometric Frobenius morphism

$$\overline{F}^*_X : H^*_{sm}(\overline{X};\mathbb{Q}_l) \to H^*_{sm}(\overline{X};\mathbb{Q}_l)$$

given as:

$$\overline{F}^*_X(1) = 1$$
$$\overline{F}^*_X([\overline{X}]) = q[\overline{X}]$$
$$\overline{F}^*_X(\alpha_i) = \lambda_i\alpha_i \ (i = 1, 2, \ldots 2g)$$

where $\lambda_i \in \overline{\mathbb{Q}}_l$ algebraic with $|\lambda_i| = q^{1/2}$ for any embedding of λ_i in $\mathbb{C}$.

II. Cohomology of moduli stacks of vector bundles

- $\mathcal{B}GL_n$ = **classifying stack** of all rank n vector bundles

$$\mathcal{B}GL_n : (Sch/\mathbb{F}_q) \to (Groupoids)$$

with $S \mapsto \mathcal{B}GL_n(S)$ = category with

II. Cohomology of moduli stacks of vector bundles

- $\mathcal{B}GL_n$ = **classifying stack** of all rank n vector bundles

$$\mathcal{B}GL_n : (Sch/\mathbb{F}_q) \to (Groupoids)$$

with $S \mapsto \mathcal{B}GL_n(S)$ = category with

  - **objects:** vector bundles $\mathcal{E} \downarrow S$ of rank n on S

II. Cohomology of moduli stacks of vector bundles

- $\mathcal{B}GL_n$ = **classifying stack** of all rank n vector bundles

$$\mathcal{B}GL_n : (Sch/\mathbb{F}_q) \to (Groupoids)$$

with $S \mapsto \mathcal{B}GL_n(S)$ = category with

  - **objects:** vector bundles $\mathcal{E} \downarrow S$ of rank n on S
  - **morphisms:** vector bundle isomorphisms

II. Cohomology of moduli stacks of vector bundles

- $\mathcal{B}GL_n$ = **classifying stack** of all rank n vector bundles

$$\mathcal{B}GL_n : (Sch/\mathbb{F}_q) \to (Groupoids)$$

with $S \mapsto \mathcal{B}GL_n(S)$ = category with

  - **objects:** vector bundles $\mathcal{E} \downarrow S$ of rank n on S
  - **morphisms:** vector bundle isomorphisms

- $\overline{\mathcal{B}GL}_n := \mathcal{B}GL_n \times_{Spec(\mathbb{F}_q)} Spec(\overline{\mathbb{F}}_q)$ with geometric Frobenius

$$\overline{F}^*_{\mathcal{B}GL_n} : H^*_{sm}(\overline{\mathcal{B}GL}_n; \mathbb{Q}_l) \to H^*_{sm}(\overline{\mathcal{B}GL}_n; \mathbb{Q}_l)$$

II. Cohomology of moduli stacks of vector bundles

- $\mathcal{B}GL_n$ = **classifying stack** of all rank n vector bundles

$$\mathcal{B}GL_n : (Sch/\mathbb{F}_q) \to (Groupoids)$$

  with $S \mapsto \mathcal{B}GL_n(S)$ = category with
  - **objects:** vector bundles $\mathcal{E} \downarrow S$ of rank n on S
  - **morphisms:** vector bundle isomorphisms
- $\overline{\mathcal{B}GL}_n := \mathcal{B}GL_n \times_{Spec(\mathbb{F}_q)} Spec(\overline{\mathbb{F}}_q)$ with geometric Frobenius

$$\overline{F}^*_{\mathcal{B}GL_n} : H^*_{sm}(\overline{\mathcal{B}GL}_n; \mathbb{Q}_l) \to H^*_{sm}(\overline{\mathcal{B}GL}_n; \mathbb{Q}_l)$$

Theorem (Behrend ('93))

There is an isomorphism of graded $\mathbb{Q}_l$-algebras

$$H^*_{sm}(\overline{\mathcal{B}GL}_n; \mathbb{Q}_l) \cong \mathbb{Q}_l[c_1, \ldots c_n]$$

and the absolute geometric Frobenius $\overline{F}^*_{\mathcal{B}GL_n}$ acts as

$$\overline{F}^*_{\mathcal{B}GL_n}(c_i) = q^i c_i \quad (i \geq 1)$$

II. Cohomology of moduli stacks of vector bundles

- universal bundle $\mathcal{E}^{univ} \downarrow \overline{X} \times \overline{Bun_X^{n,d}}$ gives morphism of stacks

$$u : \overline{X} \times \overline{Bun_X^{n,d}} \to \overline{\mathcal{B}GL_n}$$

and has **Chern classes**

$$c_i(\mathcal{E}^{univ}) = u^*(c_i) \in H^{2i}_{sm}(\overline{X} \times \overline{Bun_X^{n,d}}; \mathbb{Q}_l)$$

II. Cohomology of moduli stacks of vector bundles

- universal bundle $\mathcal{E}^{univ} \downarrow \overline{X} \times \overline{Bun}_X^{n,d}$ gives morphism of stacks

$$u : \overline{X} \times \overline{Bun}_X^{n,d} \to \overline{\mathcal{B}GL_n}$$

 and has **Chern classes**

$$c_i(\mathcal{E}^{univ}) = u^*(c_i) \in H^{2i}_{sm}(\overline{X} \times \overline{Bun}_X^{n,d}; \mathbb{Q}_l)$$

- Have **Künneth decomposition** of characteristic classes

$$c_i(\mathcal{E}^{univ}) = 1 \otimes c_i + \sum_{j=1}^{2g} \alpha_j \otimes a_i^{(j)} + [\overline{X}] \otimes b_{i-1}$$

 where $c_i \in H^{2i}_{sm}(\overline{Bun}_X^{n,d}; \mathbb{Q}_l)$, $a_i^{(j)} \in H^{2i}_{sm}(\overline{Bun}_X^{n,d}; \mathbb{Q}_l)$ and $b_{i-1} \in H^{2(i-1)}_{sm}(\overline{Bun}_X^{n,d}; \mathbb{Q}_l)$ are the **Atiyah-Bott classes.**

II. Cohomology of moduli stacks of vector bundles

Theorem (Harder-Narasimhan ('72), Atiyah-Bott ('82), N.-Stuhler ('05), Heinloth-Schmitt ('10))

There is an isomorphism of graded $\mathbb{Q}_l$-algebras

$$H^*_{sm}(\overline{Bun}_X^{n,d}; \mathbb{Q}_l) \cong \mathbb{Q}_l[c_1, \ldots, c_n, b_1, \ldots, b_n] \otimes \\ \otimes \Lambda_{\mathbb{Q}_l}(a_1^{(1)}, \ldots, a_1^{(2g)}, \ldots, a_n^{(1)}, \ldots, a_n^{(2g)})$$

II. Cohomology of moduli stacks of vector bundles

Theorem (Harder-Narasimhan ('72), Atiyah-Bott ('82), N.-Stuhler ('05), Heinloth-Schmitt ('10))

There is an isomorphism of graded $\mathbb{Q}_l$-algebras

$$H^*_{sm}(\overline{Bun}_X^{n,d}; \mathbb{Q}_l) \cong \mathbb{Q}_l[c_1, \ldots, c_n, b_1, \ldots, b_n] \otimes \otimes \Lambda_{\mathbb{Q}_l}(a_1^{(1)}, \ldots, a_1^{(2g)}, \ldots, a_n^{(1)}, \ldots, a_n^{(2g)})$$

Proof. (Steps).

(1) Show that $H^*_{sm}(\overline{Bun}_X^{n,d}; \mathbb{Q}_l)$ contains graded $\mathbb{Q}_l$-algebra of RHS via induction over rank and reduction to closed substacks of vector bundles being direct sums of line bundles.

II. Cohomology of moduli stacks of vector bundles

Theorem (Harder-Narasimhan ('72), Atiyah-Bott ('82), N.-Stuhler ('05), Heinloth-Schmitt ('10))

There is an isomorphism of graded $\mathbb{Q}_l$-algebras

$$H^*_{sm}(\overline{Bun}_X^{n,d}; \mathbb{Q}_l) \cong \mathbb{Q}_l[c_1, \ldots, c_n, b_1, \ldots, b_n] \otimes \otimes \Lambda_{\mathbb{Q}_l}(a_1^{(1)}, \ldots, a_1^{(2g)}, \ldots, a_n^{(1)}, \ldots, a_n^{(2g)})$$

Proof. (Steps).

(1) Show that $H^*_{sm}(\overline{Bun}_X^{n,d}; \mathbb{Q}_l)$ contains graded $\mathbb{Q}_l$-algebra of RHS via induction over rank and reduction to closed substacks of vector bundles being direct sums of line bundles.

(2) Calculate Poincaré series of the stack $\overline{Bun}_X^{n,d}$ by "stackifying" [Bifet-Ghione-Letizia ('94)]: $\overline{Bun}_X^{n,d}$ is quasi-isomorphic with a certain ind-scheme $\overline{Div}^{n,d}$ representing a moduli functor of effective divisors on X □

II. Cohomology of moduli stacks of vector bundles

(1) **Induction over the rank.**

Let $n = 1$. $Bun_X^{1,d}$ moduli stack of line bundles of degree d.

II. Cohomology of moduli stacks of vector bundles

(1) **Induction over the rank.**
Let $n = 1$. $Bun_X^{1,d}$ moduli stack of line bundles of degree d.

- $Bun_X^{1,d}$ has coarse moduli space Pic_X^d and we get a $\mathbb{G}_m$-gerbe $Bun_X^{1,d} \to Pic_X^d$.

II. Cohomology of moduli stacks of vector bundles

(1) **Induction over the rank.**

Let $n = 1$. $Bun_X^{1,d}$ moduli stack of line bundles of degree d.

- $Bun_X^{1,d}$ has coarse moduli space Pic_X^d and we get a $\mathbb{G}_m$-gerbe $Bun_X^{1,d} \to Pic_X^d$.
- Ramanan ('73), Drezet-Narasimhan ('89): There exists a Poincaré family on $X \times Pic_X^d$ and a splitting $Bun_X^{1,d} \cong Pic_X^d \times \mathcal{B}\mathbb{G}_m$.

II. Cohomology of moduli stacks of vector bundles

(1) **Induction over the rank.**
Let $n = 1$. $Bun_X^{1,d}$ moduli stack of line bundles of degree d.

- $Bun_X^{1,d}$ has coarse moduli space Pic_X^d and we get a $\mathbb{G}_m$-gerbe $Bun_X^{1,d} \to Pic_X^d$.
- Ramanan ('73), Drezet-Narasimhan ('89): There exists a Poincaré family on $X \times Pic_X^d$ and a splitting $Bun_X^{1,d} \cong Pic_X^d \times \mathcal{B}\mathbb{G}_m$.
- Künneth decomposition gives:

$$H^*_{sm}(\overline{Bun}_X^{1,d}, \mathbb{Q}_l) \cong H^*_{sm}(\overline{Pic}_X^d, \mathbb{Q}_l) \otimes H^*_{sm}(\overline{\mathcal{B}\mathbb{G}_m}, \mathbb{Q}_l)$$

II. Cohomology of moduli stacks of vector bundles

(1) **Induction over the rank.**

Let $n = 1$. $Bun_X^{1,d}$ moduli stack of line bundles of degree d.

- $Bun_X^{1,d}$ has coarse moduli space Pic_X^d and we get a $\mathbb{G}_m$-gerbe $Bun_X^{1,d} \to Pic_X^d$.
- Ramanan ('73), Drezet-Narasimhan ('89): There exists a Poincaré family on $X \times Pic_X^d$ and a splitting $Bun_X^{1,d} \cong Pic_X^d \times \mathcal{B}\mathbb{G}_m$.
- Künneth decomposition gives:

$$\begin{aligned} H^*_{sm}(\overline{Bun}_X^{1,d}, \mathbb{Q}_l) &\cong H^*_{sm}(\overline{Pic}_X^d, \mathbb{Q}_l) \otimes H^*_{sm}(\overline{\mathcal{B}\mathbb{G}_m}, \mathbb{Q}_l) \\ &\cong \Lambda_{\mathbb{Q}_l}(a_1^{(1)}, \ldots, a_1^{(2g)}) \otimes \mathbb{Q}_l[c_1]. \end{aligned}$$

II. Cohomology of moduli stacks of vector bundles

(1) **Induction over the rank.**

Let $n = 1$. $Bun_X^{1,d}$ moduli stack of line bundles of degree d.

- $Bun_X^{1,d}$ has coarse moduli space Pic_X^d and we get a $\mathbb{G}_m$-gerbe $Bun_X^{1,d} \to Pic_X^d$.
- Ramanan ('73), Drezet-Narasimhan ('89): There exists a Poincaré family on $X \times Pic_X^d$ and a splitting $Bun_X^{1,d} \cong Pic_X^d \times \mathcal{B}\mathbb{G}_m$.
- Künneth decomposition gives:

$$H^*_{sm}(\overline{Bun}_X^{1,d}, \mathbb{Q}_l) \cong H^*_{sm}(\overline{Pic}_X^d, \mathbb{Q}_l) \otimes H^*_{sm}(\overline{\mathcal{B}\mathbb{G}_m}, \mathbb{Q}_l)$$
$$\cong \Lambda_{\mathbb{Q}_l}(a_1^{(1)}, \ldots, a_1^{(2g)}) \otimes \mathbb{Q}_l[c_1].$$

because have

$$H^*_{sm}(\overline{Pic}_X^d, \mathbb{Q}_l) \cong H^*_{sm}(\overline{Jac(X)}, \mathbb{Q}_l) \cong \Lambda_{\mathbb{Q}_l}(H^1_{et}(\overline{X}, \mathbb{Q}_l)).$$

II. Cohomology of moduli stacks of vector bundles

Let $n > 1$. Take arbitrary partition of rank d

$$d = \sum_{i=1}^{n} d_i,\ d_i \in \mathbb{Z};\ \underline{d} = (d_1, \ldots, d_n)$$

and analyze

$$\oplus_{\underline{d}} : \prod_{i=1}^{n} Bun_X^{1,d_i} \to Bun_X^{n,d},\ (\mathcal{L}_i) \mapsto \mathcal{L}_1 \oplus \ldots \oplus \mathcal{L}_n.$$

and $(\oplus_{\underline{d}})^*$ and use that Chern classes of direct sums of line bundles are expressed as elementary symmetric polynomials.

II. Cohomology of moduli stacks of vector bundles

Let $n > 1$. Take arbitrary partition of rank d

$$d = \sum_{i=1}^{n} d_i,\ \ d_i \in \mathbb{Z};\ \ \underline{d} = (d_1, \ldots, d_n)$$

and analyze

$$\oplus_{\underline{d}} : \prod_{i=1}^{n} Bun_X^{1,d_i} \to Bun_X^{n,d},\ \ (\mathcal{L}_i) \mapsto \mathcal{L}_1 \oplus \ldots \oplus \mathcal{L}_n.$$

and $(\oplus_{\underline{d}})^*$ and use that Chern classes of direct sums of line bundles are expressed as elementary symmetric polynomials.

Get a commutative diagram

$$\begin{array}{ccc}
H^*_{sm}(\overline{Bun}_X^{n,d}; \mathbb{Q}_l) & \longrightarrow & \prod_{\underline{d}} \mathbb{Q}_l[C_1, \ldots C_n] \otimes \bigotimes_{i,j}^{n,2g} \Lambda_{\mathbb{Q}_l}(A_i^{(j)}) \\
\uparrow{\scriptstyle \alpha} & & \uparrow{\scriptstyle \psi} \\
\mathbb{Q}_l[c_i] \otimes \mathbb{Q}_l[a_i^{(j)}] \otimes \mathbb{Q}_l[b_i] & \xrightarrow{\ \varphi\ } & \mathbb{Q}_l[C_i] \otimes \mathbb{Q}_l[A_i^{(j)}] \otimes \frac{\mathbb{Q}_l[D_1, \ldots D_n]}{(\sum_s D_{s-d})}
\end{array}$$

II. Cohomology of moduli stacks of vector bundles

(2) **”Stackify” approach of Biffet-Ghione-Letizia.**

II. Cohomology of moduli stacks of vector bundles

(2) **"Stackify" approach of Biffet-Ghione-Letizia.**

- $\Lambda =$ poset of effective divisors on X. For any $D \in \Lambda$ have **moduli functor**

$$\underline{Div}^{n,d}(D) : (Sch/\mathbb{F}_q) \to (Sets)$$

  where
  $\underline{Div}^{n,d}(D)(S) =$ equivalence classes of inclusions $\mathcal{F} \hookrightarrow \mathcal{O}_{X\times S}(D)^n$ with $\mathcal{F}$ family of rank n and degree d bundles on $X \times S$

II. Cohomology of moduli stacks of vector bundles

(2) **"Stackify" approach of Biffet-Ghione-Letizia.**

- $\Lambda =$ poset of effective divisors on X. For any $D \in \Lambda$ have **moduli functor**
$$\underline{Div}^{n,d}(D) : (Sch/\mathbb{F}_q) \to (Sets)$$
where
$\underline{Div}^{n,d}(D)(S) =$ equivalence classes of inclusions $\mathcal{F} \hookrightarrow \mathcal{O}_{X\times S}(D)^n$ with $\mathcal{F}$ family of rank n and degree d bundles on $X \times S$
- $\underline{Div}^{n,d}(D)$ representable by a Quot scheme $Div^{n,d}(D)$

II. Cohomology of moduli stacks of vector bundles

(2) **"Stackify" approach of Biffet-Ghione-Letizia.**

- $\Lambda =$ poset of effective divisors on X. For any $D \in \Lambda$ have **moduli functor**
$$\underline{Div}^{n,d}(D) : (Sch/\mathbb{F}_q) \to (Sets)$$
where
$\underline{Div}^{n,d}(D)(S) =$ equivalence classes of inclusions $\mathcal{F} \hookrightarrow \mathcal{O}_{X\times S}(D)^n$ with $\mathcal{F}$ family of rank n and degree d bundles on $X \times S$
- $\underline{Div}^{n,d}(D)$ representable by a Quot scheme $Div^{n,d}(D)$
- Quot schemes $Div^{n,d}(D)$ assemble to ind-scheme $Div^{n,d}$

II. Cohomology of moduli stacks of vector bundles

(2) **"Stackify" approach of Biffet-Ghione-Letizia.**

- $\Lambda =$ poset of effective divisors on X. For any $D \in \Lambda$ have **moduli functor**
$$\underline{Div}^{n,d}(D) : (Sch/\mathbb{F}_q) \to (Sets)$$
  where
  $\underline{Div}^{n,d}(D)(S) =$ equivalence classes of inclusions $\mathcal{F} \hookrightarrow \mathcal{O}_{X\times S}(D)^n$ with $\mathcal{F}$ family of rank n and degree d bundles on $X \times S$
- $\underline{Div}^{n,d}(D)$ representable by a Quot scheme $Div^{n,d}(D)$
- Quot schemes $Div^{n,d}(D)$ assemble to ind-scheme $Div^{n,d}$
- There is a morphism of algebraic stacks
$$Div^{n,d}(D) \to Bun_X^{n,d}$$

II. Cohomology of moduli stacks of vector bundles

(2) **"Stackify" approach of Biffet-Ghione-Letizia.**

- $\Lambda =$ poset of effective divisors on X. For any $D \in \Lambda$ have **moduli functor**

$$\underline{Div}^{n,d}(D) : (Sch/\mathbb{F}_q) \to (Sets)$$

  where
  $\underline{Div}^{n,d}(D)(S) =$ equivalence classes of inclusions $\mathcal{F} \hookrightarrow \mathcal{O}_{X \times S}(D)^n$ with $\mathcal{F}$ family of rank n and degree d bundles on $X \times S$
- $\underline{Div}^{n,d}(D)$ representable by a Quot scheme $Div^{n,d}(D)$
- Quot schemes $Div^{n,d}(D)$ assemble to ind-scheme $Div^{n,d}$
- There is a morphism of algebraic stacks

$$Div^{n,d}(D) \to Bun_X^{n,d}$$

  inducing a morphism $Div^{n,d} \to Bun_X^{n,d}$

II. Cohomology of moduli stacks of vector bundles

(2) **"Stackify" approach of Biffet-Ghione-Letizia.**

- $\Lambda =$ poset of effective divisors on X. For any $D \in \Lambda$ have **moduli functor**

$$\underline{Div}^{n,d}(D) : (Sch/\mathbb{F}_q) \to (Sets)$$

  where
  $\underline{Div}^{n,d}(D)(S) =$ equivalence classes of inclusions $\mathcal{F} \hookrightarrow \mathcal{O}_{X\times S}(D)^n$ with $\mathcal{F}$ family of rank n and degree d bundles on $X \times S$
- $\underline{Div}^{n,d}(D)$ representable by a Quot scheme $Div^{n,d}(D)$
- Quot schemes $Div^{n,d}(D)$ assemble to ind-scheme $Div^{n,d}$
- There is a morphism of algebraic stacks

$$Div^{n,d}(D) \to Bun_X^{n,d}$$

  inducing a morphism $Div^{n,d} \to Bun_X^{n,d}$ and an isomorphism

$$H^i_{sm}(\overline{Bun}_X^{n,d}; \mathbb{Q}_l) \cong H^i_{sm}(\overline{Div}^{n,d}; \mathbb{Q}_l)$$

III. Frobenii and moduli stacks of vector bundles

- **Induced geometric Frobenius morphism**

  $X =$ smooth projective curve of genus g over the field $\mathbb{F}_q$

  $$F_X : (X, \mathcal{O}_X) \to (X, \mathcal{O}_X), F_X := (id_X, f \mapsto f^q)$$

  $\overline{X} = X \times_{Spec(\mathbb{F}_q)} Spec(\overline{\mathbb{F}}_q)$ base change

  $$\overline{F}_X = F_X \times id_{Spec(\overline{\mathbb{F}}_q)} : \overline{X} \to \overline{X}$$

III. Frobenii and moduli stacks of vector bundles

- **Induced geometric Frobenius morphism**

  X = smooth projective curve of genus g over the field $\mathbb{F}_q$

  $$F_X : (X, \mathcal{O}_X) \to (X, \mathcal{O}_X), F_X := (id_X, f \mapsto f^q)$$

  $\overline{X} = X \times_{Spec(\mathbb{F}_q)} Spec(\overline{\mathbb{F}}_q)$ base change

  $$\overline{F}_X = F_X \times id_{Spec(\overline{\mathbb{F}}_q)} : \overline{X} \to \overline{X}$$

  Get functor via pullback along $\overline{F}_X$

  $$\overline{Bun}_X^{n,d}(S) \to \overline{Bun}_X^{n,d}(S),\ \mathcal{E} \mapsto \overline{F}^*(\mathcal{E}) := (\overline{F}_X \times id_S)^*(\mathcal{E})$$

  inducing endomorphism of stacks

  $$\varphi : \overline{Bun}_X^{n,d} \to \overline{Bun}_X^{n,d}$$

  inducing endomorphism in cohomology

  $$\Phi = \varphi^* : H^*_{sm}(\overline{Bun}_X^{n,d}; \mathbb{Q}_l) \to H^*_{sm}(\overline{Bun}_X^{n,d}; \mathbb{Q}_l)$$

III. Frobenii and moduli stacks of vector bundles

Proposition (Naturality)

There is a canonical isomorphism

$$(\overline{F}_X \times id_{\overline{Bun}_X^{n,d}})^*(\mathcal{E}^{univ}) \cong (id_{\overline{X}} \times \varphi)^*(\mathcal{E}^{univ})$$

III. Frobenii and moduli stacks of vector bundles

Proposition (Naturality)

There is a canonical isomorphism

$$(\overline{F}_X \times id_{\overline{Bun}_X^{n,d}})^*(\mathcal{E}^{univ}) \cong (id_{\overline{X}} \times \varphi)^*(\mathcal{E}^{univ})$$

Proof.

For any stack $\mathcal{T}/\overline{\mathbb{F}}_q$ a vector bundle $\mathcal{E}$ of rank n and degree d on $\overline{X} \times \mathcal{T}$ is given by morphism of stacks

$$u : \mathcal{T} \to \overline{Bun}_X^{n,d}$$

s. th. $\mathcal{E} \cong (id_{\overline{X}} \times u)^*(\mathcal{E}^{univ})$.

Apply this to the vector bundle $(\overline{F}_X \times id_{\overline{Bun}_X^{n,d}})^*(\mathcal{E}^{univ})$. □

III. Frobenii and moduli stacks of vector bundles

Theorem (N.-Stuhler ('05), Castorena-N. ('24))

The induced geometric Frobenius $\Phi = \varphi^*$ *acts on* $H^i_{sm}(\overline{Bun_X^{n,d}}; \mathbb{Q}_l)$ *as follows:*

$$\varphi^*(c_i) = c_i \ \ (i \geq 1)$$
$$\varphi^*(a_i^{(j)}) = \lambda_j a_i^{(j)} \ \ (i \geq 1; j = 1, \ldots, 2g)$$
$$\varphi^*(b_i) = q b_i \ \ (i \geq 1)$$

III. Frobenii and moduli stacks of vector bundles

Theorem (N.-Stuhler ('05), Castorena-N. ('24))

The induced geometric Frobenius $\Phi = \varphi^*$ *acts on* $H^i_{sm}(\overline{Bun}_X^{n,d}; \mathbb{Q}_l)$ *as follows:*

$$\varphi^*(c_i) = c_i \ \ (i \geq 1)$$
$$\varphi^*(a_i^{(j)}) = \lambda_j a_i^{(j)} \ \ (i \geq 1; j = 1, \ldots, 2g)$$
$$\varphi^*(b_i) = q b_i \ \ (i \geq 1)$$

Proof. (Ingredients).

III. Frobenii and moduli stacks of vector bundles

Theorem (N.-Stuhler ('05), Castorena-N. ('24))

The induced geometric Frobenius $\Phi = \varphi^*$ *acts on* $H^i_{sm}(\overline{Bun}_X^{n,d}; \mathbb{Q}_l)$ *as follows:*

$$\varphi^*(c_i) = c_i \ \ (i \geq 1)$$
$$\varphi^*(a_i^{(j)}) = \lambda_j a_i^{(j)} \ \ (i \geq 1; j = 1, \ldots, 2g)$$
$$\varphi^*(b_i) = q b_i \ \ (i \geq 1)$$

Proof. (Ingredients).

- Use functoriality of Chern classes, i.e. get
$$(\overline{F}_X \times id_{\overline{Bun}_X^{n,d}})^*(c_i(\mathcal{E}^{univ})) = (id_{\overline{X}} \times \varphi)^*(c_i(\mathcal{E}^{univ}))$$

III. Frobenii and moduli stacks of vector bundles

Theorem (N.-Stuhler ('05), Castorena-N. ('24))

The induced geometric Frobenius $\Phi = \varphi^*$ *acts on* $H^i_{sm}(\overline{Bun}_X^{n,d}; \mathbb{Q}_l)$ *as follows:*

$$\varphi^*(c_i) = c_i \ \ (i \geq 1)$$
$$\varphi^*(a_i^{(j)}) = \lambda_j a_i^{(j)} \ \ (i \geq 1; j = 1, \ldots, 2g)$$
$$\varphi^*(b_i) = q b_i \ \ (i \geq 1)$$

Proof. (Ingredients).

- Use functoriality of Chern classes, i.e. get
$$(\overline{F}_X \times id_{\overline{Bun}_X^{n,d}})^*(c_i(\mathcal{E}^{univ})) = (id_{\overline{X}} \times \varphi)^*(c_i(\mathcal{E}^{univ}))$$
- Use Künneth decomposition for $c_i(\mathcal{E}^{univ})$, structure of $H^*_{sm}(\overline{Bun}_X^{n,d}; \mathbb{Q}_l)$ and action of geometric Frobenius $\overline{F}_X^*$ on $H^*_{sm}(\overline{X}; \mathbb{Q}_l)$ to evaluate the expressions on both sides.

□

III. Frobenii and moduli stacks of vector bundles

- **Absolute geometric Frobenius morphism**
  $(Bun_X^{n,d}, \mathcal{O}_{Bun_X^{n,d}})$ algebraic stack with structure sheaf
  Get endomorphism of stacks

$$F_{Bun_X^{n,d}} : (Bun_X^{n,d}, \mathcal{O}_{Bun_X^{n,d}}) \to (Bun_X^{n,d}, \mathcal{O}_{Bun_X^{n,d}})$$

  and its base change extension

$$\overline{F}_{Bun_X^{n,d}} := F_{Bun_X^{n,d}} \times id_{Spec(\overline{\mathbb{F}}_q)}$$

  inducing endomorphism in cohomology

$$\overline{F}^*_{Bun_X^{n,d}} : H^*_{sm}(\overline{Bun}_X^{n,d}; \mathbb{Q}_l) \to H^*_{sm}(\overline{Bun}_X^{n,d}; \mathbb{Q}_l)$$

III. Frobenii and moduli stacks of vector bundles

- **Arithmetic Frobenius morphism**
  Classical Frobenius morphism, i. e. generator of the Galois group $Gal(\overline{\mathbb{F}}_q/\mathbb{F}_q)$

$$Frob : \overline{\mathbb{F}}_q \to \overline{\mathbb{F}}_q, \;\; a \mapsto a^q$$

  Get endomorphism of schemes

$$Frob_{Spec(\overline{\mathbb{F}}_q)} : Spec(\overline{\mathbb{F}}_q) \to Spec(\overline{\mathbb{F}}_q)$$

  inducing endomorphism of stacks

$$\psi := id_{Bun_X^{n,d}} \times Frob_{Spec(\overline{\mathbb{F}}_q)} : \overline{Bun}_X^{n,d} \to \overline{Bun}_X^{n,d}$$

  inducing endomorphism in cohomology

$$\Psi = \psi^* : H^*_{sm}(\overline{Bun}_X^{n,d}; \mathbb{Q}_l) \to H^*_{sm}(\overline{Bun}_X^{n,d}; \mathbb{Q}_l)$$

III. Frobenii and moduli stacks of vector bundles

Theorem (N.-Stuhler ('05), Castorena-N. ('24))

The absolute geometric Frobenius $\overline{F}^*_{Bun_X^{n,d}}$ acts on $H^i_{sm}(\overline{Bun_X^{n,d}}; \mathbb{Q}_l)$ as follows:

$$\overline{F}^*_{Bun_X^{n,d}}(c_i) = q^i c_i \ \ (i \geq 1)$$

$$\overline{F}^*_{Bun_X^{n,d}}(a_i^{(j)}) = \lambda_j^{-1} q^i a_i^{(j)} \ \ (i \geq 1; j = 1, \ldots, 2g)$$

$$\overline{F}^*_{Bun_X^{n,d}}(b_i) = q^{i-1} b_i \ \ (i \geq 1)$$

III. Frobenii and moduli stacks of vector bundles

Theorem (N.-Stuhler ('05), Castorena-N. ('24))

The absolute geometric Frobenius $\overline{F}^*_{Bun_X^{n,d}}$ acts on $H^i_{sm}(\overline{Bun_X^{n,d}}; \mathbb{Q}_l)$ as follows:

$$\overline{F}^*_{Bun_X^{n,d}}(c_i) = q^i c_i \ \ (i \geq 1)$$
$$\overline{F}^*_{Bun_X^{n,d}}(a_i^{(j)}) = \lambda_j^{-1} q^i a_i^{(j)} \ \ (i \geq 1; j = 1, \ldots, 2g)$$
$$\overline{F}^*_{Bun_X^{n,d}}(b_i) = q^{i-1} b_i \ \ (i \geq 1)$$

Proof. (Ingredients).

- $\tilde{\mathcal{E}}^{univ} \downarrow BGL_n$ = universal bundle
  $\mathcal{E}^{univ} \downarrow X \times Bun_X^{n,d}$ = universal bundle with classifying morphism
  $u : X \times Bun_X^{n,d} \to BGL_n$ with $u^*(\tilde{\mathcal{E}}^{univ}) \cong \mathcal{E}^{univ}$
  $$F_{BGL_n} \circ u = u \circ F_{X \times Bun_X^{n,d}}$$
  $$\overline{F}^*_{X \times Bun_X^{n,d}}(c_i(\overline{u}^*(\tilde{\mathcal{E}}^{univ}))) = (\overline{F}_X \times id_{\overline{Bun_X^{n,d}}})^*(id_{\overline{X}} \times \overline{F}_{Bun_X^{n,d}})^*(c_i(\mathcal{E}^{univ}))$$

□

III. Frobenii and moduli stacks of vector bundles

Theorem (N.-Stuhler ('05), Castorena-N. ('24))

The absolute arithmetic Frobenius $\Psi = \psi^*$ *acts on* $H^i_{sm}(\overline{Bun}_X^{n,d}; \mathbb{Q}_l)$ *as follows:*

$$\psi^*(c_i) = q^{-i} c_i \quad (i \geq 1)$$
$$\psi^*(a_i^{(j)}) = \lambda_j q^{-i} a_i^{(j)} \quad (i \geq 1; j = 1, \dots, 2g)$$
$$\psi^*(b_i) = q^{-i+1} b_i \quad (i \geq 1)$$

III. Frobenii and moduli stacks of vector bundles

Theorem (N.-Stuhler ('05), Castorena-N. ('24))

The absolute arithmetic Frobenius $\Psi = \psi^*$ *acts on* $H^i_{sm}(\overline{Bun}_X^{n,d}; \mathbb{Q}_l)$ *as follows:*

$$\psi^*(c_i) = q^{-i} c_i \ \ (i \geq 1)$$
$$\psi^*(a_i^{(j)}) = \lambda_j q^{-i} a_i^{(j)} \ \ (i \geq 1; j = 1, \ldots, 2g)$$
$$\psi^*(b_i) = q^{-i+1} b_i \ \ (i \geq 1)$$

Proof. (Ingredients).

This follows from the calculations for the absolute geometric Frobenius as the absolute arithmetic Frobenius morphism ψ^* is inverse to the absolute geometric Frobenius morphism $\overline{F}^*_{Bun_X^{n,d}}$. □

III. Frobenii and moduli stacks of vector bundles

Definition

The **cardinality** of a groupoid $\mathcal{G} = [\mathcal{G}_1 \rightrightarrows \mathcal{G}_0]$ is the real number

$$\#\mathcal{G} = \sum_{x \in [\mathcal{G}]} \frac{1}{\# Aut_{\mathcal{G}}(x)},$$

where the sum is taken over isomorphism classes of objects x of $\mathcal{G}$ and $\# Aut_{\mathcal{G}}(x)$ is the order of the automorphism group of the object x. If this sum diverges we say $\#\mathcal{G} = \infty$.

III. Frobenii and moduli stacks of vector bundles

Definition

The **cardinality** of a groupoid $\mathcal{G} = [\mathcal{G}_1 \rightrightarrows \mathcal{G}_0]$ is the real number

$$\#\mathcal{G} = \sum_{x \in [\mathcal{G}]} \frac{1}{\#Aut_{\mathcal{G}}(x)},$$

where the sum is taken over isomorphism classes of objects x of $\mathcal{G}$ and $\#Aut_{\mathcal{G}}(x)$ is the order of the automorphism group of the object x. If this sum diverges we say $\#\mathcal{G} = \infty$.

Example: Let X be a finite set and $\mathcal{G} = [X \rightrightarrows X]$ be the associated groupoid. Then:

$$\#\mathcal{G} = \sum_{x \in X} 1 = \#X.$$

III. Frobenii and moduli stacks of vector bundles

Definition

The **cardinality** of a groupoid $\mathcal{G} = [\mathcal{G}_1 \rightrightarrows \mathcal{G}_0]$ is the real number

$$\#\mathcal{G} = \sum_{x \in [\mathcal{G}]} \frac{1}{\#Aut_{\mathcal{G}}(x)},$$

where the sum is taken over isomorphism classes of objects x of $\mathcal{G}$ and $\#Aut_{\mathcal{G}}(x)$ is the order of the automorphism group of the object x. If this sum diverges we say $\#\mathcal{G} = \infty$.

Example: Let X be a finite set and $\mathcal{G} = [X \rightrightarrows X]$ be the associated groupoid. Then:

$$\#\mathcal{G} = \sum_{x \in X} 1 = \#X.$$

Example: Let G be a finite group and $\mathcal{B}G = [G \rightrightarrows *]$ be the associated groupoid. Then:

$$\#\mathcal{B}G = \frac{1}{\#G}.$$

III. Frobenii and moduli stacks of vector bundles

Example: Let a finite group G act on a finite set X and $X//G = [G \times X \rightrightarrows X]$ be the action groupoid. We have:

$$\#X//G = \sum_{x \in [X//G]} \frac{1}{\#Aut_{X//G}(x)} = \frac{\#X}{\#G}.$$

III. Frobenii and moduli stacks of vector bundles

Example: Let a finite group G act on a finite set X and $X//G = [G \times X \rightrightarrows X]$ be the action groupoid. We have:

$$\#X//G = \sum_{x \in [X//G]} \frac{1}{\#Aut_{X//G}(x)} = \frac{\#X}{\#G}.$$

We have $\mathcal{B}G = [*//G]$. $X//G$ is a good replacement for the quotient X/G when the action is **not** free!

III. Frobenii and moduli stacks of vector bundles

Example: Let a finite group G act on a finite set X and $X//G = [G \times X \rightrightarrows X]$ be the action groupoid. We have:

$$\#X//G = \sum_{x \in [X//G]} \frac{1}{\#Aut_{X//G}(x)} = \frac{\#X}{\#G}.$$

We have $\mathcal{B}G = [*//G]$. $X//G$ is a good replacement for the quotient X/G when the action is **not** free!

Example: Let $\mathcal{G} = FinSets$ be the groupoid of all finite sets and bijections. Then:

$$\#\mathcal{G} = \sum_{x \in [\mathcal{G}]} \frac{1}{\#Aut_{\mathcal{G}}(x)} = \sum_{n \in \mathbb{N}_0} \frac{1}{\#S_n} = \sum_{n \in \mathbb{N}_0} \frac{1}{n!} = e,$$

as all sets with n elements are isomorphic, so the connected components correspond bijectively to the natural numbers $\mathbb{N}_0$.

III. Frobenii and moduli stacks of vector bundles

Example: Let a finite group G act on a finite set X and $X//G = [G \times X \rightrightarrows X]$ be the action groupoid. We have:

$$\#X//G = \sum_{x \in [X//G]} \frac{1}{\#Aut_{X//G}(x)} = \frac{\#X}{\#G}.$$

We have $\mathcal{B}G = [*//G]$. $X//G$ is a good replacement for the quotient X/G when the action is **not** free!

Example: Let $\mathcal{G} = FinSets$ be the groupoid of all finite sets and bijections. Then:

$$\#\mathcal{G} = \sum_{x \in [\mathcal{G}]} \frac{1}{\#Aut_{\mathcal{G}}(x)} = \sum_{n \in \mathbb{N}_0} \frac{1}{\#S_n} = \sum_{n \in \mathbb{N}_0} \frac{1}{n!} = e,$$

as all sets with n elements are isomorphic, so the connected components correspond bijectively to the natural numbers $\mathbb{N}_0$.

Exercise: Given a real number $r \in \mathbb{R}$ construct a groupoid $\mathcal{G}$ with groupoid cardinality $\#\mathcal{G} = r$.

III. Frobenii and moduli stacks of vector bundles

- $\mathfrak{X} =$ algebraic stack over Sch/S, U object of Sch/S. Let $[\mathfrak{X}(U)]$ be the **set of isomorphism classes** of objects in the groupoid $\mathfrak{X}(U)$ and the groupoid cardinality

$$\#\mathfrak{X}(U) := \sum_{x \in [\mathfrak{X}(U)]} \frac{1}{\#Aut_{\mathfrak{X}(U)}(x)}$$

III. Frobenii and moduli stacks of vector bundles

- $\mathfrak{X} =$ algebraic stack over Sch/S, U object of Sch/S. Let $[\mathfrak{X}(U)]$ be the **set of isomorphism classes** of objects in the groupoid $\mathfrak{X}(U)$ and the groupoid cardinality

$$\#\mathfrak{X}(U) := \sum_{x \in [\mathfrak{X}(U)]} \frac{1}{\# Aut_{\mathfrak{X}(U)}(x)}$$

- $X/\mathfrak{X} =$ atlas of $\mathfrak{X}$ i. e. representable smooth surjective morphism $x : X \to \mathfrak{X}$. The **dimension** of $\mathfrak{X}$ is defined as

$$\dim(\mathfrak{X}) = \dim(X) - \mathrm{rel.}\dim(X/\mathfrak{X})$$

with $\mathrm{rel.}\dim(X/\mathfrak{X}) =$ dimenson of fibers of $X \times_{\mathfrak{X}} Y \to Y$ for any $Y \to \mathfrak{X}$

III. Frobenii and moduli stacks of vector bundles

- $\mathfrak{X}$ = algebraic stack over Sch/S, U object of Sch/S. Let $[\mathfrak{X}(U)]$ be the **set of isomorphism classes** of objects in the groupoid $\mathfrak{X}(U)$ and the groupoid cardinality

$$\#\mathfrak{X}(U) := \sum_{x \in [\mathfrak{X}(U)]} \frac{1}{\# Aut_{\mathfrak{X}(U)}(x)}$$

- $X/\mathfrak{X}$ = atlas of $\mathfrak{X}$ i. e. representable smooth surjective morphism $x : X \to \mathfrak{X}$. The **dimension** of $\mathfrak{X}$ is defined as

$$\dim(\mathfrak{X}) = \dim(X) - \mathrm{rel.}\dim(X/\mathfrak{X})$$

with $\mathrm{rel.}\dim(X/\mathfrak{X})$ = dimenson of fibers of $X \times_{\mathfrak{X}} Y \to Y$ for any $Y \to \mathfrak{X}$

- **Example.** $[X/G]$ =**quotient stack**, then
$$\dim([X/G]) = \dim(X) - \dim(G)$$
$\mathcal{B}G$ = **classifying stack**, then $\dim(\mathcal{B}G) = -\dim(G)$
$Bun_X^{n,d}$ = **moduli stack of bundles**, then $\dim(Bun_X^{n,d}) = n^2(g-1)$

III. Frobenii and moduli stacks of vector bundles

Theorem (Lefschetz trace formula (Behrend '03))

Let $\mathfrak{X}$ be a smooth algebraic stack and Ψ be the arithmetic Frobenius, then have

$$q^{\dim(\mathfrak{X})} \sum_{p \geq 0} (-1)^p tr(\Psi | H^p_{sm}(\overline{\mathfrak{X}}; \mathbb{Q}_l)) = \sum_{x \in [\mathfrak{X}(\mathrm{Spec}(\mathbb{F}_q))]} \frac{1}{\# \mathrm{Aut}_{\mathfrak{X}(\mathrm{Spec}(\mathbb{F}_q))}(x)}$$

III. Frobenii and moduli stacks of vector bundles

Theorem (Lefschetz trace formula (Behrend '03))

Let $\mathfrak{X}$ be a smooth algebraic stack and Ψ be the arithmetic Frobenius, then have

$$q^{\dim(\mathfrak{X})}\sum_{p\geq 0}(-1)^p tr(\Psi|H^p_{sm}(\overline{\mathfrak{X}};\mathbb{Q}_l)) = \sum_{x\in[\mathfrak{X}(\mathrm{Spec}(\mathbb{F}_q))]}\frac{1}{\#\,\mathrm{Aut}_{\mathfrak{X}(\mathrm{Spec}(\mathbb{F}_q))}(x)}$$

Here

$$\sum_{x\in[\mathfrak{X}(\mathrm{Spec}(\mathbb{F}_q))]}\frac{1}{\#Aut_{\mathfrak{X}(\mathrm{Spec}(\mathbb{F}_q))}(x)} = \#\mathfrak{X}(\mathrm{Spec}(\mathbb{F}_q))$$

is the number of $\mathbb{F}_q$-rational points of the algebraic stack $\mathfrak{X}$, where $\#Aut_{\mathfrak{X}(\mathrm{Spec}(\mathbb{F}_q))}(x)$ is the order of the group of automorphisms of the isomorphism class x. It is the groupoid cardinality of the groupoid $\mathfrak{X}(\mathbb{F}_q) = \mathfrak{X}(\mathrm{Spec}(\mathbb{F}_q))$ of $\mathbb{F}_q$-rational points of the stack $\mathfrak{X}$.

III. Frobenii and moduli stacks of vector bundles

Example. $\mathcal{B}\mathbb{G}_m =$ **classifying stack of line bundles**

The quotient morphism $\pi : \mathbb{A}^n - \{0\} \to \mathbb{P}^{n-1}$ is a principal $\mathbb{G}_m$-bundle

$$\begin{array}{ccc}
\mathbb{A}^n - \{0\} & \longrightarrow & \mathrm{Spec}(\mathbb{F}_q) \\
\downarrow{\scriptstyle \pi} & & \downarrow \\
\mathbb{P}^{n-1} & \xrightarrow{\phi} & \mathcal{B}\mathbb{G}_m
\end{array}$$

III. Frobenii and moduli stacks of vector bundles

Example. $\mathcal{B}\mathbb{G}_m =$ **classifying stack of line bundles**

The quotient morphism $\pi : \mathbb{A}^n - \{0\} \to \mathbb{P}^{n-1}$ is a principal $\mathbb{G}_m$-bundle

$$\begin{array}{ccc} \mathbb{A}^n - \{0\} & \longrightarrow & \mathrm{Spec}(\mathbb{F}_q) \\ \downarrow{\scriptstyle \pi} & & \downarrow \\ \mathbb{P}^{n-1} & \xrightarrow{\ \phi\ } & \mathcal{B}\mathbb{G}_m \end{array}$$

The fiber of π is $\mathbb{A}^n - \{0\}$ and have Leray spectral sequence

$$E_2^{p,q} \cong H^p_{sm}(\overline{\mathcal{B}\mathbb{G}_m}, R^q\phi_*\mathbb{Q}_l) \Rightarrow H^*_{sm}(\overline{\mathbb{P}}^{n-1}, \mathbb{Q}_l)$$

and because $R^0\phi_*\mathbb{Q}_l \cong \mathbb{Q}_l$ and $R^q\phi_*\mathbb{Q}_l = 0$ if $q < 2n-1$ it follows for $q < 2n-1$ that

$$H^q_{sm}(\overline{\mathcal{B}\mathbb{G}_m}, \mathbb{Q}_l) \cong H^q_{sm}(\overline{\mathbb{P}}^{n-1}, \mathbb{Q}_l)$$

and therefore

III. Frobenii and moduli stacks of vector bundles

Example. $\mathcal{B}\mathbb{G}_m =$ **classifying stack of line bundles**

The quotient morphism $\pi : \mathbb{A}^n - \{0\} \to \mathbb{P}^{n-1}$ is a principal $\mathbb{G}_m$-bundle

$$\begin{array}{ccc} \mathbb{A}^n - \{0\} & \longrightarrow & \mathrm{Spec}(\mathbb{F}_q) \\ \downarrow{\scriptstyle \pi} & & \downarrow \\ \mathbb{P}^{n-1} & \xrightarrow{\ \phi\ } & \mathcal{B}\mathbb{G}_m \end{array}$$

The fiber of π is $\mathbb{A}^n - \{0\}$ and have Leray spectral sequence

$$E_2^{p,q} \cong H^p_{sm}(\overline{\mathcal{B}\mathbb{G}_m}, R^q\phi_*\mathbb{Q}_l) \Rightarrow H^*_{sm}(\overline{\mathbb{P}}^{n-1}, \mathbb{Q}_l)$$

and because $R^0\phi_*\mathbb{Q}_l \cong \mathbb{Q}_l$ and $R^q\phi_*\mathbb{Q}_l = 0$ if $q < 2n-1$ it follows for $q < 2n-1$ that

$$H^q_{sm}(\overline{\mathcal{B}\mathbb{G}_m}, \mathbb{Q}_l) \cong H^q_{sm}(\overline{\mathbb{P}}^{n-1}, \mathbb{Q}_l)$$

and therefore

$$H^*_{sm}(\overline{\mathcal{B}\mathbb{G}_m}, \mathbb{Q}_l) \cong \mathbb{Q}_l[c_1]$$

where c_1 is generator of degree 2 given as Chern class of the universal line bundle $\mathcal{L}^{univ}$ on $\mathcal{B}\mathbb{G}_m$.

III. Frobenii and moduli stacks of vector bundles

- $\dim(\mathcal{B}\mathbb{G}_m) = -\dim(\mathbb{G}_m) = -1$

III. Frobenii and moduli stacks of vector bundles

- $\dim(\mathcal{B}\mathbb{G}_m) = -\dim(\mathbb{G}_m) = -1$
- $\#\mathcal{B}\mathbb{G}_m(\mathbb{F}_q) = \#$line bundles (up to iso) over "point" $\mathrm{Spec}(\mathbb{F}_q)$

III. Frobenii and moduli stacks of vector bundles

- $\dim(\mathcal{B}\mathbb{G}_m) = -\dim(\mathbb{G}_m) = -1$
- $\#\mathcal{B}\mathbb{G}_m(\mathbb{F}_q) = \#$line bundles (up to iso) over "point" $\mathrm{Spec}(\mathbb{F}_q)$
- all line bundles over the "point" $\mathrm{Spec}(\mathbb{F}_q)$ are trivial, i.e. there is just one iso class x in $\mathcal{B}\mathbb{G}_m(\mathrm{Spec}(\mathbb{F}_q))$

III. Frobenii and moduli stacks of vector bundles

- $\dim(\mathcal{B}\mathbb{G}_m) = -\dim(\mathbb{G}_m) = -1$
- $\#\mathcal{B}\mathbb{G}_m(\mathbb{F}_q) = \#$line bundles (up to iso) over "point" $\mathrm{Spec}(\mathbb{F}_q)$
- all line bundles over the "point" $\mathrm{Spec}(\mathbb{F}_q)$ are trivial, i.e. there is just one iso class x in $\mathcal{B}\mathbb{G}_m(\mathrm{Spec}(\mathbb{F}_q))$
- $\#Aut_{\mathcal{B}\mathbb{G}_m(\mathrm{Spec}(\mathbb{F}_q))}(x) = \#\mathbb{G}_m(\mathbb{F}_q) = \#\mathbb{F}_q^* = q-1$ i.e.

$$\#\mathcal{B}\mathbb{G}_m(\mathbb{F}_q) = \sum_{x \in [\mathcal{B}\mathbb{G}_m(\mathrm{Spec}(\mathbb{F}_q))]} \frac{1}{\#Aut_{\mathcal{B}\mathbb{G}_m(\mathrm{Spec}(\mathbb{F}_q))}(x)} = \frac{1}{q-1}.$$

III. Frobenii and moduli stacks of vector bundles

- $\dim(\mathcal{B}\mathbb{G}_m) = -\dim(\mathbb{G}_m) = -1$
- $\#\mathcal{B}\mathbb{G}_m(\mathbb{F}_q) = \#$line bundles (up to iso) over "point" $\mathrm{Spec}(\mathbb{F}_q)$
- all line bundles over the "point" $\mathrm{Spec}(\mathbb{F}_q)$ are trivial, i.e. there is just one iso class x in $\mathcal{B}\mathbb{G}_m(\mathrm{Spec}(\mathbb{F}_q))$
- $\#Aut_{\mathcal{B}\mathbb{G}_m(\mathrm{Spec}(\mathbb{F}_q))}(x) = \#\mathbb{G}_m(\mathbb{F}_q) = \#\mathbb{F}_q^* = q-1$ i.e.

$$\#\mathcal{B}\mathbb{G}_m(\mathbb{F}_q) = \sum_{x\in[\mathcal{B}\mathbb{G}_m(\mathrm{Spec}(\mathbb{F}_q))]} \frac{1}{\#Aut_{\mathcal{B}\mathbb{G}_m(\mathrm{Spec}(\mathbb{F}_q))}(x)} = \frac{1}{q-1}.$$

- cohomology of $\mathcal{B}\mathbb{G}_m$ is cohomology of "infinite projective space", i.e.

$$q^{\dim(\mathcal{B}\mathbb{G}_m)} \sum_{i\geq 0} tr(\Psi | H^{2i}_{sm}(\overline{\mathcal{B}\mathbb{G}_m}; \mathbb{Q}_l)) = \frac{1}{q}\sum_{i=0}^{\infty}\frac{1}{q^i}$$

III. Frobenii and moduli stacks of vector bundles

- $\dim(\mathcal{B}\mathbb{G}_m) = -\dim(\mathbb{G}_m) = -1$
- $\#\mathcal{B}\mathbb{G}_m(\mathbb{F}_q) = \#$line bundles (up to iso) over "point" $\mathrm{Spec}(\mathbb{F}_q)$
- all line bundles over the "point" $\mathrm{Spec}(\mathbb{F}_q)$ are trivial, i.e. there is just one iso class x in $\mathcal{B}\mathbb{G}_m(\mathrm{Spec}(\mathbb{F}_q))$
- $\#Aut_{\mathcal{B}\mathbb{G}_m(\mathrm{Spec}(\mathbb{F}_q))}(x) = \#\mathbb{G}_m(\mathbb{F}_q) = \#\mathbb{F}_q^* = q-1$ i.e.

$$\#\mathcal{B}\mathbb{G}_m(\mathbb{F}_q) = \sum_{x\in[\mathcal{B}\mathbb{G}_m(\mathrm{Spec}(\mathbb{F}_q))]} \frac{1}{\#Aut_{\mathcal{B}\mathbb{G}_m(\mathrm{Spec}(\mathbb{F}_q))}(x)} = \frac{1}{q-1}.$$

- cohomology of $\mathcal{B}\mathbb{G}_m$ is cohomology of "infinite projective space", i.e.

$$q^{\dim(\mathcal{B}\mathbb{G}_m)} \sum_{i\geq 0} tr(\Psi|H^{2i}_{sm}(\overline{\mathcal{B}\mathbb{G}_m}; \mathbb{Q}_l)) = \frac{1}{q}\sum_{i=0}^{\infty}\frac{1}{q^i}$$

- so get well known formula via "stacky" proof

$$\sum_{i=0}^{\infty}\frac{1}{q^{i+1}} = \frac{1}{q-1}$$

III. Frobenii and moduli stacks of vector bundles

For the zeta function of $\mathcal{B}\mathbb{G}_m$ we therefore get:

$$Z_{\mathcal{B}\mathbb{G}_m}(t) =$$

III. Frobenii and moduli stacks of vector bundles

For the zeta function of $\mathcal{B}\mathbb{G}_m$ we therefore get:

$$Z_{\mathcal{B}\mathbb{G}_m}(t) = \exp\left(\sum_{i=1}^{\infty} \#\mathcal{B}\mathbb{G}_m(\mathbb{F}_{q^i})\frac{t^i}{i}\right)$$

III. Frobenii and moduli stacks of vector bundles

For the zeta function of $\mathcal{B}\mathbb{G}_m$ we therefore get:

$$Z_{\mathcal{B}\mathbb{G}_m}(t) = \exp\Big(\sum_{i=1}^{\infty} \#\mathcal{B}\mathbb{G}_m(\mathbb{F}_{q^i})\frac{t^i}{i}\Big) = \exp\Big(\sum_{i=1}^{\infty} \frac{1}{q^i - 1}\frac{t^i}{i}\Big)$$

III. Frobenii and moduli stacks of vector bundles

For the zeta function of $\mathcal{B}\mathbb{G}_m$ we therefore get:

$$Z_{\mathcal{B}\mathbb{G}_m}(t) = \exp\left(\sum_{i=1}^{\infty} \#\mathcal{B}\mathbb{G}_m(\mathbb{F}_{q^i})\frac{t^i}{i}\right) = \exp\left(\sum_{i=1}^{\infty} \frac{1}{q^i - 1}\frac{t^i}{i}\right)$$

$$= \exp\left(\sum_{i=1}^{\infty} \frac{t^i}{i} \sum_{k=1}^{\infty} \frac{1}{q^{ki}}\right)$$

III. Frobenii and moduli stacks of vector bundles

For the zeta function of $\mathcal{B}\mathbb{G}_m$ we therefore get:

$$Z_{\mathcal{B}\mathbb{G}_m}(t) = \exp(\sum_{i=1}^{\infty} \#\mathcal{B}\mathbb{G}_m(\mathbb{F}_{q^i})\frac{t^i}{i}) = \exp(\sum_{i=1}^{\infty} \frac{1}{q^i - 1}\frac{t^i}{i})$$

$$= \exp(\sum_{i=1}^{\infty} \frac{t^i}{i} \sum_{k=1}^{\infty} \frac{1}{q^{ki}}) = \prod_{k=1}^{\infty} \exp(\sum_{i=1}^{\infty} \frac{(t/q^k)^i}{i})$$

III. Frobenii and moduli stacks of vector bundles

For the zeta function of $\mathcal{B}\mathbb{G}_m$ we therefore get:

$$Z_{\mathcal{B}\mathbb{G}_m}(t) = \exp(\sum_{i=1}^{\infty} \#\mathcal{B}\mathbb{G}_m(\mathbb{F}_{q^i})\frac{t^i}{i}) = \exp(\sum_{i=1}^{\infty} \frac{1}{q^i - 1}\frac{t^i}{i})$$

$$= \exp(\sum_{i=1}^{\infty} \frac{t^i}{i} \sum_{k=1}^{\infty} \frac{1}{q^{ki}}) = \prod_{k=1}^{\infty} \exp(\sum_{i=1}^{\infty} \frac{(t/q^k)^i}{i})$$

$$= \prod_{k=1}^{\infty} (1 - q^{-k}t)^{-1}.$$

III. Frobenii and moduli stacks of vector bundles

For the zeta function of $\mathcal{B}\mathbb{G}_m$ we therefore get:

$$\begin{aligned} Z_{\mathcal{B}\mathbb{G}_m}(t) = & \exp(\sum_{i=1}^{\infty} \#\mathcal{B}\mathbb{G}_m(\mathbb{F}_{q^i})\frac{t^i}{i}) = \exp(\sum_{i=1}^{\infty} \frac{1}{q^i - 1}\frac{t^i}{i}) \\ & = \exp(\sum_{i=1}^{\infty} \frac{t^i}{i} \sum_{k=1}^{\infty} \frac{1}{q^{ki}}) = \prod_{k=1}^{\infty} \exp(\sum_{i=1}^{\infty} \frac{(t/q^k)^i}{i}) \\ & = \prod_{k=1}^{\infty} (1 - q^{-k} t)^{-1}. \end{aligned}$$

$Z_{\mathcal{B}\mathbb{G}_m}(t)$ has a meromorphic continuation to the complex plane with simple poles at $t = q^k$ for $k \geq 1$.

III. Frobenii and moduli stacks of vector bundles

For the zeta function of $\mathcal{B}\mathbb{G}_m$ we therefore get:

$$
\begin{aligned}
Z_{\mathcal{B}\mathbb{G}_m}(t) &= \exp(\sum_{i=1}^{\infty} \#\mathcal{B}\mathbb{G}_m(\mathbb{F}_{q^i})\frac{t^i}{i}) = \exp(\sum_{i=1}^{\infty} \frac{1}{q^i - 1}\frac{t^i}{i}) \\
&= \exp(\sum_{i=1}^{\infty} \frac{t^i}{i} \sum_{k=1}^{\infty} \frac{1}{q^{ki}}) = \prod_{k=1}^{\infty} \exp(\sum_{i=1}^{\infty} \frac{(t/q^k)^i}{i}) \\
&= \prod_{k=1}^{\infty} (1 - q^{-k}t)^{-1}.
\end{aligned}
$$

$Z_{\mathcal{B}\mathbb{G}_m}(t)$ has a meromorphic continuation to the complex plane with simple poles at $t = q^k$ for $k \geq 1$.

Exercise: Calculate the zeta function for the classifying stack $\mathcal{B}GL_n$ of all rank n vector bundles!

III. Frobenii and moduli stacks of vector bundles

Theorem (Weil Conjectures, Part I, N. ('24))

Let X be a smooth projective curve of genus g over $\mathbb{F}_q$, α_i the eigenvalues of the geometric Frobenius on $H^1_{sm}(\overline{X}; \mathbb{Q}_l)$. Then:

III. Frobenii and moduli stacks of vector bundles

Theorem (Weil Conjectures, Part I, N. ('24))

Let X be a smooth projective curve of genus g over $\mathbb{F}_q$, α_i the eigenvalues of the geometric Frobenius on $H^1_{sm}(\overline{X}; \mathbb{Q}_l)$. Then:

(1) *The number of $\mathbb{F}_q$-rational points of $Bun_X^{n,d}$ is*

$$\#Bun_X^{n,d}(\mathbb{F}_q) = q^{n^2(g-1)} \frac{\prod_{i=1}^{n} \prod_{j=1}^{2g} (1 - \alpha_j q^{-i})}{\prod_{i=1}^{n} (1 - q^{-i}) \prod_{i=2}^{n} (1 - q^{-i+1})}$$

III. Frobenii and moduli stacks of vector bundles

Theorem (Weil Conjectures, Part I, N. ('24))

Let X be a smooth projective curve of genus g over $\mathbb{F}_q$, α_i the eigenvalues of the geometric Frobenius on $H^1_{sm}(\overline{X};\mathbb{Q}_l)$. Then:

(1) *The number of $\mathbb{F}_q$-rational points of $Bun_X^{n,d}$ is*

$$\#Bun_X^{n,d}(\mathbb{F}_q) = q^{n^2(g-1)} \frac{\prod_{i=1}^{n}\prod_{j=1}^{2g}(1-\alpha_j q^{-i})}{\prod_{i=1}^{n}(1-q^{-i})\prod_{i=2}^{n}(1-q^{-i+1})}$$

(2) *The $\zeta-$ function $Z_{Bun_X^{n,d}}(t) = \exp(\sum_{i=1}^{\infty}\#Bun_X^{n,d}(\mathbb{F}_{q^i})\frac{t^i}{i})$ is a meromorphic function with convergent product expansion*

$$Z_{Bun_X^{n,d}}(t) = \prod_{i=1}^{\infty} det(1-\Psi q^{\dim(Bun_X^{n,d})} t | H^i_{sm}(\overline{Bun}_X^{n,d};\mathbb{Q}_l))^{(-1)^{i+1}}$$

III. Frobenii and moduli stacks of vector bundles

Proof. (Ingredients).

(1) is variation of a calculation by [Harder-Narasimhan ('72)] using **Lefschetz trace formula** of arithmetic Frobenius Ψ for algebraic stacks $\mathfrak{X}$ [Behrend ('03)]

$$q^{dim(\mathfrak{X})} \sum_{p \geq 0} (-1)^p tr(\Psi | H^p_{sm}(\overline{\mathfrak{X}}; \mathbb{Q}_l)) = \sum_{x \in [\mathfrak{X}(\mathbb{F}_q)]} \frac{1}{\# Aut_{\mathfrak{X}(\mathbb{F}_q)}(x)}$$

where for $\mathfrak{X} = Bun^{n,d}_X$ have $dim(Bun^{n,d}_X) = n^2(g-1)$

III. Frobenii and moduli stacks of vector bundles

Proof. (Ingredients).

(1) is variation of a calculation by [Harder-Narasimhan ('72)] using **Lefschetz trace formula** of arithmetic Frobenius Ψ for algebraic stacks $\mathfrak{X}$ [Behrend ('03)]

$$q^{dim(\mathfrak{X})} \sum_{p \geq 0} (-1)^p tr(\Psi | H^p_{sm}(\overline{\mathfrak{X}}; \mathbb{Q}_l)) = \sum_{x \in [\mathfrak{X}(\mathbb{F}_q)]} \frac{1}{\# Aut_{\mathfrak{X}(\mathbb{F}_q)}(x)}$$

where for $\mathfrak{X} = Bun_X^{n,d}$ have $dim(Bun_X^{n,d}) = n^2(g-1)$

(2) Product expansion of ζ-function is proved by [Behrend ('93)] for algebraic stacks $\mathfrak{X}$ of finite type using Lefschetz trace formula.

III. Frobenii and moduli stacks of vector bundles

Proof. (Ingredients).

(1) is variation of a calculation by [Harder-Narasimhan ('72)] using **Lefschetz trace formula** of arithmetic Frobenius Ψ for algebraic stacks $\mathfrak{X}$ [Behrend ('03)]

$$q^{dim(\mathfrak{X})}\sum_{p\geq 0}(-1)^p tr(\Psi|H^p_{sm}(\overline{\mathfrak{X}};\mathbb{Q}_l)) = \sum_{x\in[\mathfrak{X}(\mathbb{F}_q)]}\frac{1}{\#Aut_{\mathfrak{X}(\mathbb{F}_q)}(x)}$$

where for $\mathfrak{X} = Bun_X^{n,d}$ have $dim(Bun_X^{n,d}) = n^2(g-1)$

(2) Product expansion of ζ-function is proved by [Behrend ('93)] for algebraic stacks $\mathfrak{X}$ of finite type using Lefschetz trace formula.

- $Bun_X^{n,d,\leq p} \subset Bun_X^{n,d}$: open substack with $Bun_X^{n,d,\leq p}(S) =$ groupoid of vector bundles $\mathcal{E} \downarrow X \times S$ of rank n and degree d with Shatz polygon $sh(\mathcal{E}|X\times s) \leq p$ for all closed points s of S.

III. Frobenii and moduli stacks of vector bundles

Proof. (Ingredients).

(2) (contin.)

III. Frobenii and moduli stacks of vector bundles

Proof. (Ingredients).

(2) (contin.)

- $Bun_X^{n,d,<p} \subset Bun_X^{n,d}$: open substack

III. Frobenii and moduli stacks of vector bundles

Proof. (Ingredients).

(2) (contin.)

- $Bun_X^{n,d,<p} \subset Bun_X^{n,d}$: open substack
- $Bun_X^{n,d,p} \subset Bun_X^{n,d,\leq p} \backslash Bun_X^{n,d,<p}$: closed substack of finite type

III. Frobenii and moduli stacks of vector bundles

Proof. (Ingredients).

(2) (contin.)

- $Bun_X^{n,d,<p} \subset Bun_X^{n,d}$: open substack
- $Bun_X^{n,d,p} \subset Bun_X^{n,d,\leq p} \backslash Bun_X^{n,d,<p}$: closed substack of finite type
- Gysin sequence for closed embedding $Bun_X^{n,d,p} \subset Bun_X^{n,d,\leq p}$ splits into short exact sequences

$$0 \to H_{sm}^{*-2m}(\overline{Bun}_X^{n,d,p}, \mathbb{Q}_l(m)) \to H_{sm}^{*}(\overline{Bun}_X^{n,d,\leq p}, \mathbb{Q}_l)$$

$$\to H^{*}(\overline{Bun}_X^{n,d,<p}, \mathbb{Q}_l) \to 0$$

and Lefschetz trace formula gives:

$$q^{dim(Bun_X^{n,d})} tr(\Psi | H_{sm}^{*}(\overline{Bun}_X^{n,d}; \mathbb{Q}_l)) = \sum_p q^{dim(Bun_X^{n,d,p})} tr(\Psi | H_{sm}^{*}(\overline{Bun}_X^{n,d,p}; \mathbb{Q}_l))$$

□

III. Frobenii and moduli stacks of vector bundles

Theorem (Weil Conjectures, Part II, N. ('24))

(3) *The eigenvalues of the arithmetic Frobenius* Ψ *acting on* $H^i_{sm}(\overline{Bun}^{n,d}_X; \mathbb{Q}_l)$ *have absolute value* $q^{i/2}$ *and the Poincaré series of* $H^*_{sm}(\overline{Bun}^{n,d}_X; \mathbb{Q}_l)$ *is given as:*

$$P_{Bun^{n,d}_X}(t) = \frac{\prod_{i=1}^n (1+t^{2i-1})^{2g}}{\prod_{i=1}^n (1-t^{2i}) \prod_{i=2}^n (1-t^{2i-2})}$$

Proof. (Ingredients).

(3) uses reduction to "nice" covering substacks of $Bun^{n,d}_X$ looking locally like quotient stacks $[Z/\mathbb{G}_m]$ having the desired properties for the arithmetic Frobenius Ψ and Leray spectral sequence

$$E_2^{p,q} \cong H^p_{sm}(\mathcal{B}\mathbb{G}_m; \mathbb{Q}_l) \otimes H^q_{sm}(Z, \mathbb{Q}_l) \Rightarrow H^{p+q}_{sm}([Z/\mathbb{G}_m]; \mathbb{Q}_l).$$

□

III. Frobenii and moduli stacks of vector bundles

Proposition

The formal trace

$$tr(\Phi^r \times \Psi^s | H^*_{sm}(\overline{Bun}_X^{n,d}; \mathbb{Q}_l)) = \sum_{p \geq 0} (-1)^p tr(\Phi^r \times \Psi^s | H^p_{sm}(\overline{Bun}_X^{n,d}; \mathbb{Q}_l))$$

is absolutely convergent for $s > r$.

III. Frobenii and moduli stacks of vector bundles

Proposition

The formal trace

$$tr(\Phi^r \times \Psi^s | H^*_{sm}(\overline{Bun}_X^{n,d}; \mathbb{Q}_l)) = \sum_{p \geq 0} (-1)^p tr(\Phi^r \times \Psi^s | H^p_{sm}(\overline{Bun}_X^{n,d}; \mathbb{Q}_l))$$

is absolutely convergent for $s > r$.

Proof.

The formal trace is given by:

$$(\prod_{i=1}^{n}(\sum_{n=0}^{\infty} q^{-is})) \cdot (\prod_{k=1}^{n-1}(\sum_{m=0}^{\infty} q^{m(r-ks)})) \cdot (\prod_{j=1}^{2g}\prod_{i=1}^{n}(1 + |\lambda_j|^{r+s} q^{-is})$$

□

III. Frobenii and moduli stacks of vector bundles

- for $r = 0$, $s = 1$ there is **Lefschetz Trace Formula** of [Behrend]

III. Frobenii and moduli stacks of vector bundles

- for $r=0$, $s=1$ there is **Lefschetz Trace Formula** of [Behrend]
- consider $Bun^{2,0}_{\mathbb{P}^1}$ i. e. rank 2 bundles on $\mathbb{P}^1$ with trivial determinant

$$H^*_{sm}(\overline{Bun}^{2,0}_{\mathbb{P}^1};\mathbb{Q}_l) \cong \mathbb{Q}_l[c_2, b_1]$$

with $c_2 \in H^4_{sm}(\overline{Bun}^{2,0}_{\mathbb{P}^1};\mathbb{Q}_l)$, $b_1 \in H^2_{sm}(\overline{Bun}^{2,0}_{\mathbb{P}^1};\mathbb{Q}_l)$. Then:

$$tr(\Phi^r \times \Psi^s | H^*_{sm}(\overline{Bun}^{2,0}_{\mathbb{P}^1};\mathbb{Q}_l) = (\sum_{m=0}^{\infty} q^{-2sm}) \cdot (\sum_{m=0}^{\infty} q^{(r-s)m})$$

absolutely convergent for $s > r$, i. e. for $s > r$

$$tr(\Phi^r \times \Psi^s | H^*_{sm}(\overline{Bun}^{2,0}_{\mathbb{P}^1};\mathbb{Q}_l) = (1-q^{-2s})^{-1} \cdot (1-q^{r-s})^{-1}$$

III. Frobenii and moduli stacks of vector bundles

- Fixed points of $\Phi^r \times \Psi^s$

$$(\overline{Bun}^{2,0}_{\mathbb{P}^1})^{\Phi^r \times \Psi^s} = (\overline{Bun^{ss}}^{2,0}_{\mathbb{P}^1})^{\Phi^r \times \Psi^s} = (\overline{\mathcal{B}SL_2})^{\Psi^s}$$

where $Bun^{ss\,2,0}_{\mathbb{P}^1}$ is the open substack of semistable bundles

III. Frobenii and moduli stacks of vector bundles

- Fixed points of $\Phi^r \times \Psi^s$

$$(\overline{Bun}^{2,0}_{\mathbb{P}^1})^{\Phi^r \times \Psi^s} = (\overline{Bun^{ss}}^{2,0}_{\mathbb{P}^1})^{\Phi^r \times \Psi^s} = (\overline{\mathcal{B}SL_2})^{\Psi^s}$$

  where $Bun^{ss\,2,0}_{\mathbb{P}^1}$ is the open substack of semistable bundles
- Fixed point stack does **not** depend on r, so naive fixed point formula fails!

III. Frobenii and moduli stacks of vector bundles

- Fixed points of $\Phi^r \times \Psi^s$

$$(\overline{Bun}_{\mathbb{P}^1}^{2,0})^{\Phi^r \times \Psi^s} = (\overline{Bun^{ss}}_{\mathbb{P}^1}^{2,0})^{\Phi^r \times \Psi^s} = (\overline{\mathcal{B}SL_2})^{\Psi^s}$$

  where $Bun^{ss}{}_{\mathbb{P}^1}^{2,0}$ is the open substack of semistable bundles
- Fixed point stack does **not** depend on r, so naive fixed point formula fails!

Open Questions.

III. Frobenii and moduli stacks of vector bundles

- Fixed points of $\Phi^r \times \Psi^s$

$$(\overline{Bun}_{\mathbb{P}^1}^{2,0})^{\Phi^r \times \Psi^s} = (\overline{Bun^{ss}}_{\mathbb{P}^1}^{2,0})^{\Phi^r \times \Psi^s} = (\overline{\mathcal{B}SL_2})^{\Psi^s}$$

  where $Bun^{ss}{}_{\mathbb{P}^1}^{2,0}$ is the open substack of semistable bundles
- Fixed point stack does **not** depend on r, so naive fixed point formula fails!

Open Questions.

- Is there a modified trace formula for the composition $\Phi^r \times \Psi^s$ if $s > r$? What does it ”count” geometrically?

III. Frobenii and moduli stacks of vector bundles

- Fixed points of $\Phi^r \times \Psi^s$

$$(\overline{Bun}^{2,0}_{\mathbb{P}^1})^{\Phi^r \times \Psi^s} = (\overline{Bun^{ss}}^{2,0}_{\mathbb{P}^1})^{\Phi^r \times \Psi^s} = (\overline{\mathcal{B}SL_2})^{\Psi^s}$$

  where $Bun^{ss}{}^{2,0}_{\mathbb{P}^1}$ is the open substack of semistable bundles
- Fixed point stack does **not** depend on r, so naive fixed point formula fails!

Open Questions.

- Is there a modified trace formula for the composition $\Phi^r \times \Psi^s$ if $s > r$? What does it "count" geometrically?
- Are there analogs of the Weil Conjectures for the geometric Frobenii Φ and $\overline{F}^*_{Bun^{n,d}_X}$?

III. Frobenii and moduli stacks of vector bundles

- Fixed points of $\Phi^r \times \Psi^s$

$$(\overline{Bun}^{2,0}_{\mathbb{P}^1})^{\Phi^r \times \Psi^s} = (\overline{Bun^{ss}}^{2,0}_{\mathbb{P}^1})^{\Phi^r \times \Psi^s} = (\overline{\mathcal{B}SL_2})^{\Psi^s}$$

  where $Bun^{ss\,2,0}_{\mathbb{P}^1}$ is the open substack of semistable bundles
- Fixed point stack does **not** depend on r, so naive fixed point formula fails!

Open Questions.

- Is there a modified trace formula for the composition $\Phi^r \times \Psi^s$ if $s > r$? What does it "count" geometrically?
- Are there analogs of the Weil Conjectures for the geometric Frobenii Φ and $\overline{F}^*_{Bun^{n,d}_X}$?
- Are there analogs of the Weil Conjectures for the absolute Frobenii Φ or $F_{\mathfrak{X}}$ on general algebraic stacks $\mathfrak{X}$ over $\mathbb{F}_q$ (for algebraic stacks smooth, locally of finite type...)? (work in progress, see also work by Sun (2010))

References

- F. Neumann, Weil Conjectures for moduli stacks of principal bundles over algebraic curves. in preparation.
- F. Neumann, Geometry of Algebraic Stacks and Moduli of Vector Bundles. in preparation for *IMPA Research Monographs*, Springer Verlag, Berlin-New York, 300 pp.
- A. Castorena, F. Neumann, On actions of Frobenius morphisms for moduli stacks of principal bundles over algebraic curves. *Bulletin des Sciences Mathématiques* 193 (2024), 103450.
- F. Neumann, U. Stuhler, Moduli stacks of vector bundles and Frobenius morphisms. *Algebra and Number Theory, Proc. of the Inter. Conf.*, HIN/AMS Delhi, 2005, 126–146.